

Time Series Analysis

Spring 2024

Peder Bacher and Pernille Yde Nielsen February 2, 2024

Outline of the lecture

- Welcome to the course
 - Practical information
 - Homepage + Ed discussion
 - Evaluation
- Introduction to time series analysis (Chapter 1)
 - Examples of time series data
- Multivariate random variables (Chapter 2)
 - Probability distributions
 - The multivariate normal distribution
 - Moment representation



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Einer Ari Árnason



Peter Grønning



Dimitrios Sousounis









Course homepage

https://02417.compute.dtu.dk/

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DTU 2017 Overview Solutions					
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Basic course information					
Contact					
• Teachers: Peder Bacher Pernille Yde Nielsen					
DTU Learn					
Course page on DTU Learn. Messages will be posted there and assignments will be available there.					
Locations					
Lectures: Fridays 8-10. Location: Building 208 - Auditorium 54					
 Exercises: Fridays 10-12. Location: Building 210 - Rooms 012, 112, 118, 142, 148 and 162 (40 seated ach) 	s in				
Reading and exercise material					
See Overview					
Assignments					
Four assignments must be carried out, more information will be given.					

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Ed Discussion

https://edstem.org/eu/join/BRq6NW

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- 4 Assignments
- Peergrade

Introductory example – COLO B shares, 1 month

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Introductory example – COLO B shares, 1 year

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Introductory example – COLO B shares, all

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#### Number of monthly airline passengers in the US



### Consumption of district heating (VEKS) data



# Consumption of district heating – static model



#### Consumption of district heating – model error



#### At the end of the course I want you to be able to



Mink and Muskrat skins traded

Years

# Make a forecast

Mink and Muskrat skins traded in Canada 1850–1911



Years

# And confidence intervals



Years

# Multivariate random variables

- What is a random variable?
- What is a multivariate random variable?
- Why are multivariate random variables essential in time series analysis?







#### Multivariate random variables

Definition (n-dimensional random variable; random vector)

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix}$$

Joint distribution function:

$$F(x_1,\ldots,x_n) = P\{X_1 \leq x_1,\ldots,X_n \leq x_n\}$$

When does this simplify to

$$F(x_1,...,x_n) = P\{X_1 \le x_1\} P\{X_2 \le x_2\} ... P\{X_n \le x_n\}?$$

#### Joint distribution function and joint density function

Joint distribution function (repeated from last slide):

$$F(x_1,\ldots,x_n) = P\{X_1 \leq x_1,\ldots,X_n \leq x_n\}$$

Joint density function (often abbreviated **pdf**):

$$f(x_1,\ldots,x_n)=\frac{\partial^n F(x_1,\ldots,x_n)}{\partial x_1\ldots\partial x_n}$$

(and back to the distribution function:)

$$F(x_1,\ldots,x_n)=\int_{-\infty}^{x_1}\ldots\int_{-\infty}^{x_n}f(t_1,\ldots,t_n)\,dt_1\ldots\,dt_n$$

#### The multivariate normal distribution

Joint pdf (probability density function):

$$f_{\boldsymbol{X}}(\boldsymbol{x}) = \frac{1}{(2\pi)^{n/2}\sqrt{\det \boldsymbol{\Sigma}}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{T}\boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right]$$

**S** is the **variance-covariance matrix** (symmetric)

If the variables are uncorrelated then the variance-covariance matrix  $\Sigma = \begin{pmatrix} \sigma_1^2 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sigma_p^2 \end{pmatrix}$ 

Notation:  $\pmb{X} \sim N(\pmb{\mu}, \pmb{\Sigma})$  means the multivariate random variable  $\pmb{X}$  is normally distributed



2-dimensional example

# Joint and marginal density function



### Marginal density function

When we only consider part of the variables.

**X** is a random vector of size n. We consider 'sub-vector' of size k (k < n).

Marginal density function:  $f_S(x_1, \ldots, x_k) = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f(x_1, \ldots, x_n) dx_{k+1} \ldots dx_n$ Example in 2D: 0.40.2  $^{0}_{-5}$ 5 0 () 5 -5

# Conditional density function

When part of the variables are fixed.



# Conditional density function

When part of the variables are fixed.

Conditional density function:

$$f_{X_1|X_2=x_2}(x_1) = \frac{f_{X_1,X_2}(x_1,x_2)}{f_{X_2}(x_2)}$$



# Independence

If variables X and Y are independent, then:  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ 

If X and Y are independent, then the distribution of Y does not depend on X:

$$f_{Y|X=x}(y) = f_Y(y)$$

Independence implies no correlation - not the other way around





E[X] = the **expectation value** is the same as the average of unrealized random variables.

$$E[X] = \int_{-\infty}^{\infty} x \, f_X(x) \, dx$$

E[..] is also called the "expectation operator"

Expectation is a **linear operator**:

$$E[a + bX_1 + cX_2] = a + bE[X_1] + cE[X_2]$$

# Moments and central moments

n'th moment:

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) \, dx$$

n'th central moment:

$$E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - E[X])^n f_X(x) \, dx$$

### Variance and covariance

The variance is the same as the 2nd central moment:

$$V[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

Covariance is a *mixed* central moment:

 $Cov[X_1, X_2] = E[(X_1 - E[X_1])(X_2 - E[X_2])] = E[X_1X_2] - E[X_1]E[X_2]$ 

 $V[X] = \operatorname{Cov}[X, X]$ 

#### Moments of random vectors

Expectation:  $E[X] = [E[X_1], E[X_2], ..., E[X_n]]^T$ 

Variance-covariance matrix:

$$\Sigma_{\boldsymbol{X}} = V[\boldsymbol{X}] = E[(\boldsymbol{X} - \boldsymbol{\mu})(\boldsymbol{X} - \boldsymbol{\mu})^{T}] = \begin{bmatrix} V[X_{1}] & \text{Cov}[X_{1}, X_{2}] & \cdots & \text{Cov}[X_{1}, X_{n}] \\ \text{Cov}[X_{2}, X_{1}] & V[X_{2}] & \cdots & \text{Cov}[X_{2}, X_{n}] \\ \vdots & & \vdots \\ \text{Cov}[X_{n}, X_{1}] & \text{Cov}[X_{n}, X_{2}] & \cdots & V[X_{n}] \end{bmatrix}$$

Correlation: 
$$\rho_{ij} = \frac{\text{Cov}[X_i, X_j]}{\sqrt{V[X_i]V[X_j]}} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$$

## Calculation rule

Calculation rule worth remembering (2.28):

 $Cov[aX_1 + bX_2, cX_3 + dX_4] =$  $ac Cov[X_1, X_3] + ad Cov[X_1, X_4] + bc Cov[X_2, X_3] + bd Cov[X_2, X_4]$ 

The rule can be used for variance as well:

$$V[a + bX_2] = b^2 V[X_2]$$
  
$$V[aX_1 + bX_2] = a^2 V[X_1] + b^2 V[X_2] + 2ab Cov[X_1, X_2]$$

### Moment representation

With the "moment representation" we describe all the moments up to a certain order (but we do not describe the full probability distribution).

"Second order moment representation" consists of:

- Mean (expectation)
- Variance
- Covariance (if relevant)

The normal distribution is fully characterized by its second order representation.

#### Conditional expectations

Conditional expectation: 
$$E[$$

$$\mathbf{E}[Y|X=x] = \int_{-\infty}^{\infty} y f_{Y|X=x}(y) \, dy$$

Conditional variance:  $\operatorname{Var}[\boldsymbol{Y}|\boldsymbol{X}] = \operatorname{E}[(\boldsymbol{Y} - \operatorname{E}[\boldsymbol{Y}|\boldsymbol{X}])(\boldsymbol{Y} - \operatorname{E}[\boldsymbol{Y}|\boldsymbol{X}])^T|\boldsymbol{X}]$ 

Variance separation theorem ((2.51) and (2.52) in the book):

$$Var[\boldsymbol{Y}] = E[Var[\boldsymbol{Y}|\boldsymbol{X}]] + Var[E[\boldsymbol{Y}|\boldsymbol{X}]]$$
$$C[\boldsymbol{Y}, \boldsymbol{Z}] = E[C[\boldsymbol{Y}, \boldsymbol{Z}|\boldsymbol{X}]] + C[E[\boldsymbol{Y}|\boldsymbol{X}], E[\boldsymbol{Z}|\boldsymbol{X}]]$$

Obs: in the last equation C[..] is the same as Cov[..]



Exercise 2.1, 2.2, 2.3

Use (2.25) (2.28) (2.39) (2.40) (2.51) and (2.52) from the book

Obs: Ex 2.2 should say "independent" instead of "uncorrelated"