Time Series Analysis

Week 12 –

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Week 12: Outline of the lecture

Recursive and adaptive estimation:

- Recursive LS, Section 11.1
- Recursive pseudo-linear regression, Section 11.2
- Model-based adaptive estimation, Section 11.4

Why recursive and adaptive estimation?

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- New information should be included by “adjustment” rather than recalculating everything
- Models are approximations
- The best approximation may change over time
- Makes it possible to produce software which learns as new data becomes available

RLS – Types of models considered

REG:

$$Y_t = \mu + \beta_1 U_{1,t} + \beta_2 U_{2,t} + \ldots + \beta_m U_{m,t} + \varepsilon_t$$

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AR:

$$\begin{aligned} \phi(B) Y_t &= \mu + \varepsilon_t \Leftrightarrow \\ Y_t &= \mu - \phi_1 Y_{t-1} - \phi_2 Y_{t-2} - \ldots - \phi_p Y_{t-p} + \varepsilon_t \end{aligned}$$

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FIR:

$$Y_t = \mu + \omega(B) U_t + \varepsilon_t$$

$$= \mu + \omega_0 U_t + \omega_1 U_{t-1} + + \ldots \omega_s U_{t-s} + \varepsilon_t$$

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Generic form of the models considered

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Example:

$$Y_t = \mu \cdot \underbrace{1}_{x_{1,t}} + \phi_2 \cdot \underbrace{(-Y_{t-2})}_{x_{2,t}} + \omega_1 \cdot \underbrace{U_{t-1}}_{x_{3,t}} + \varepsilon_t$$

LS-estimate at time t

Model:

$$Y_t = \boldsymbol{x}_t^T \boldsymbol{\theta} + \varepsilon_t$$

Data ($\boldsymbol{x}$ may contain lagged values of the "real" input/output):

$$Y_1, Y_2, Y_3, Y_4, \ldots, Y_{t-1}, Y_t$$

$$\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_4, \boldsymbol{x}_4, \ldots, \boldsymbol{x}_{t-1}, \boldsymbol{x}_t$$

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LS-estimate based on t observations:

$$S_t(\boldsymbol{\theta}) = \sum_{s=1}^{t} (Y_s - \boldsymbol{x}_s^T \boldsymbol{\theta})^2$$

$$\widehat{\boldsymbol{\theta}}_t = \arg\min_{\theta} S_t(\boldsymbol{\theta})$$

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From one time step to the next (in an easy way)

The trick is to realize that:

$$\boldsymbol{R}_t = \boldsymbol{X}^T \boldsymbol{X} = \boldsymbol{x}_1 \boldsymbol{x}_1^T + \boldsymbol{x}_2 \boldsymbol{x}_2^T + \ldots + \boldsymbol{x}_t \boldsymbol{x}_t^T = \sum_{s=1}^{t} \boldsymbol{x}_s \boldsymbol{x}_s^T = \sum_{s=1}^{t-1} \boldsymbol{x}_s \boldsymbol{x}_s^T + \boldsymbol{x}_t \boldsymbol{x}_t^T$$

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Where:

$$\boldsymbol{x}_t \boldsymbol{x}_t^T = \begin{bmatrix} x_{1,t} x_{1,t} & x_{1,t} x_{2,t} & \cdots & x_{1,t} x_{\ell,t} \\ x_{2,t} x_{1,t} & x_{2,t} x_{2,t} & \cdots & x_{2,t} x_{\ell,t} \\ \vdots & \vdots & \ddots & \vdots \\ x_{\ell,t} x_{1,t} & x_{\ell,t} x_{2,t} & \cdots & x_{\ell,t} x_{\ell,t} \end{bmatrix} \qquad \boldsymbol{x}_t Y_t = \begin{bmatrix} x_{1,t} Y_t \\ x_{2,t} Y_t \\ \vdots \\ x_{\ell,t} Y_t \end{bmatrix}$$

The RLS algorithm

$$\widehat{\boldsymbol{\theta}}_t = \boldsymbol{R}_t^{-1} \boldsymbol{h}_t$$

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Initialization:

- $\boldsymbol{R}_0 = \mathbf{0}$ (matrix of zeros)
- $\boldsymbol{h}_0 = \mathbf{0}$ (vector of zeros)
- Start estimating $\widehat{\boldsymbol{\theta}}_t$ when $\boldsymbol{R}_t$ is invertible

The RLS algorithm – 2 equivalent formulations

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$$\boldsymbol{R}_t = \boldsymbol{R}_{t-1} + \boldsymbol{x}_t \boldsymbol{x}_t^T$$
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2. Where $\boldsymbol{P}_t = \boldsymbol{R}_t^{-1}$:

$$\boldsymbol{P}_t = \boldsymbol{P}_{t-1} - \frac{\boldsymbol{P}_{t-1} \boldsymbol{x}_t}{1 + \boldsymbol{x}_t^T \boldsymbol{P}_{t-1} \boldsymbol{x}_t} \boldsymbol{x}_t^T \boldsymbol{P}_{t-1}$$
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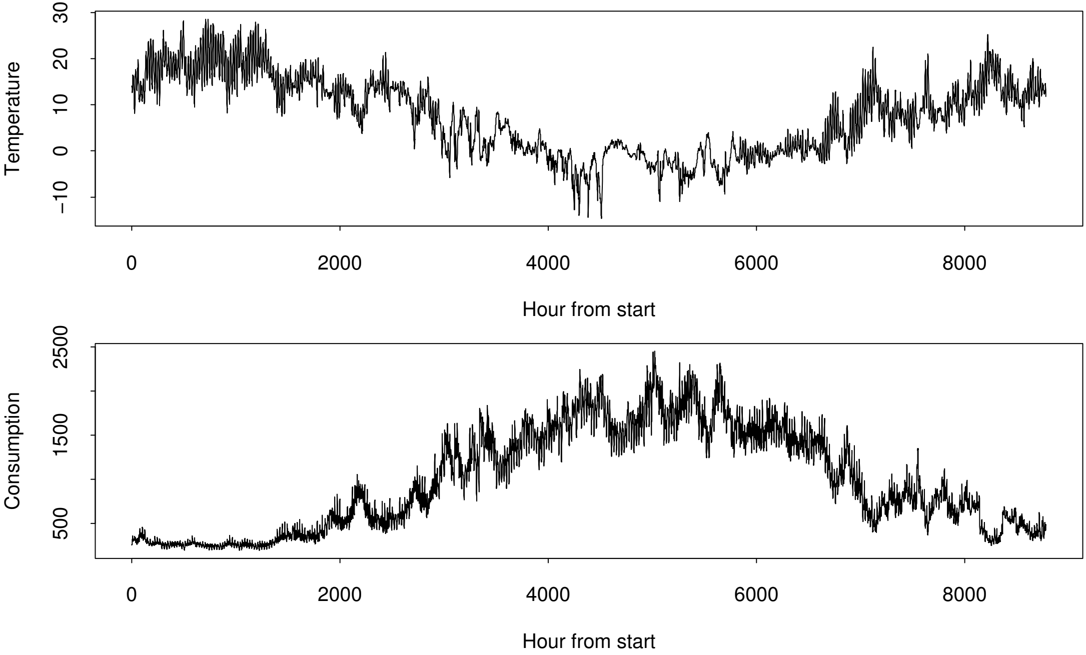
Which one would you prefer for applications and why? Second expression avoids matrix-inversion.

Example: $HC_t = \mu + \theta_1 T_t + \varepsilon_t$



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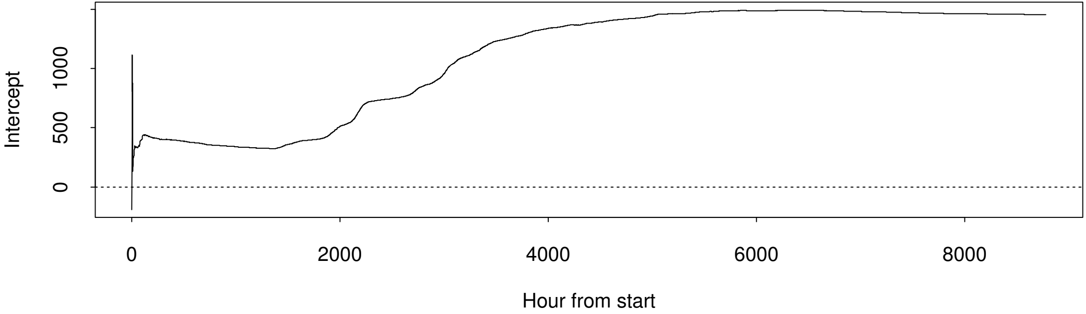

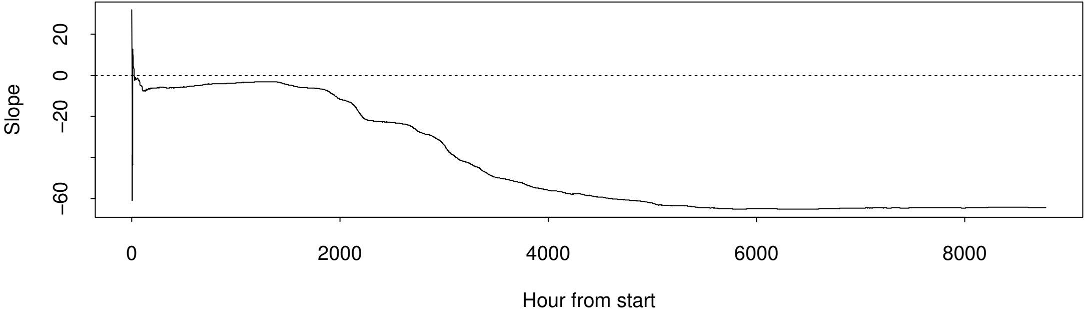

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$$S_t(\boldsymbol{\theta}) = \sum_{s=1}^{t} \beta(t, s)(Y_s - \boldsymbol{x}_s^T \boldsymbol{\theta})^2$$

where $\boldsymbol{W} = \text{diag}(\beta(t, 1), \beta(t, 2), \ldots, \beta(t, t-1), 1)$

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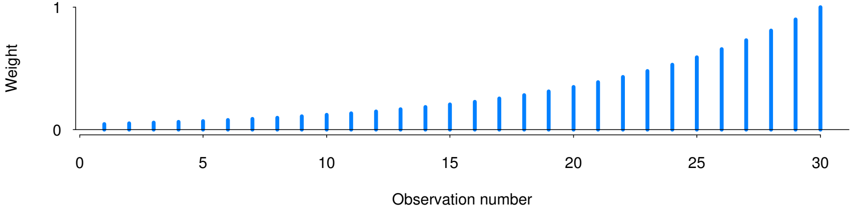
- $\beta(t, s)$ expresses how we assign weights to old observations. What does $\beta(t, s)$ usually look like?

Exponential decay of weights

- Let's first consider $\beta(t, s) = \lambda^{t-s}$ $(0 < \lambda \leq 1)$

  $\lambda = 1$: What we did with the previous algorithms

  $0 < \lambda < 1$: We "forget" in an exponential manner



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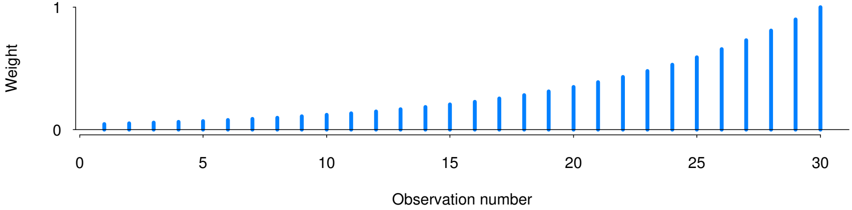

- In the general case it turns out that if the sequence of weights can be written

$$\beta(t, s) = \lambda(t)\beta(t-1, s) \qquad 1 \le s \le t-1$$
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Then the estimates can be updated recursively.

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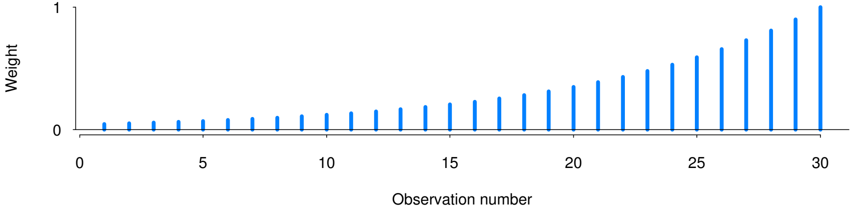

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Then the estimates can be updated recursively. Interpret the limitations these equations impose.

The Adaptive Recursive LS algorithm

$$
\begin{aligned}
\boldsymbol{R}_t &= \boldsymbol{x}_t \boldsymbol{x}_t^T + \lambda(t) \boldsymbol{R}_{t-1} \\
\boldsymbol{h}_t &= \boldsymbol{x}_t Y_t + \lambda(t) \boldsymbol{h}_{t-1} \\
\widehat{\boldsymbol{\theta}}_t &= \boldsymbol{R}_t^{-1} \boldsymbol{h}_t
\end{aligned}
$$

The Adaptive RLS algorithm – 2 equivalent formulations

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2. Eliminating $\boldsymbol{h}_t$ and avoiding matrix-inversion:

$$\boldsymbol{K}_t = \frac{\boldsymbol{P}_{t-1}\boldsymbol{x}_t}{\lambda(t) + \boldsymbol{x}_t^T\boldsymbol{P}_{t-1}\boldsymbol{x}_t}$$
$$\widehat{\boldsymbol{\theta}}_t = \widehat{\boldsymbol{\theta}}_{t-1} + \boldsymbol{K}_t(Y_t - \boldsymbol{x}_t^T\widehat{\boldsymbol{\theta}}_{t-1})$$
$$\boldsymbol{P}_t = \frac{1}{\lambda(t)}\left(\boldsymbol{P}_{t-1} - \frac{\boldsymbol{P}_{t-1}\boldsymbol{x}_t\boldsymbol{x}_t^T\boldsymbol{P}_{t-1}}{\lambda(t) + \boldsymbol{x}_t^T\boldsymbol{P}_{t-1}\boldsymbol{x}_t}\right)$$

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- Given a data set an optimal value of λ can be found by "trial and error"
- Instead of a linear grid of λ's, use a linear grid of T_0's
- Important detail: The initial estimates might be poor. How should this be taken into account when choosing λ?

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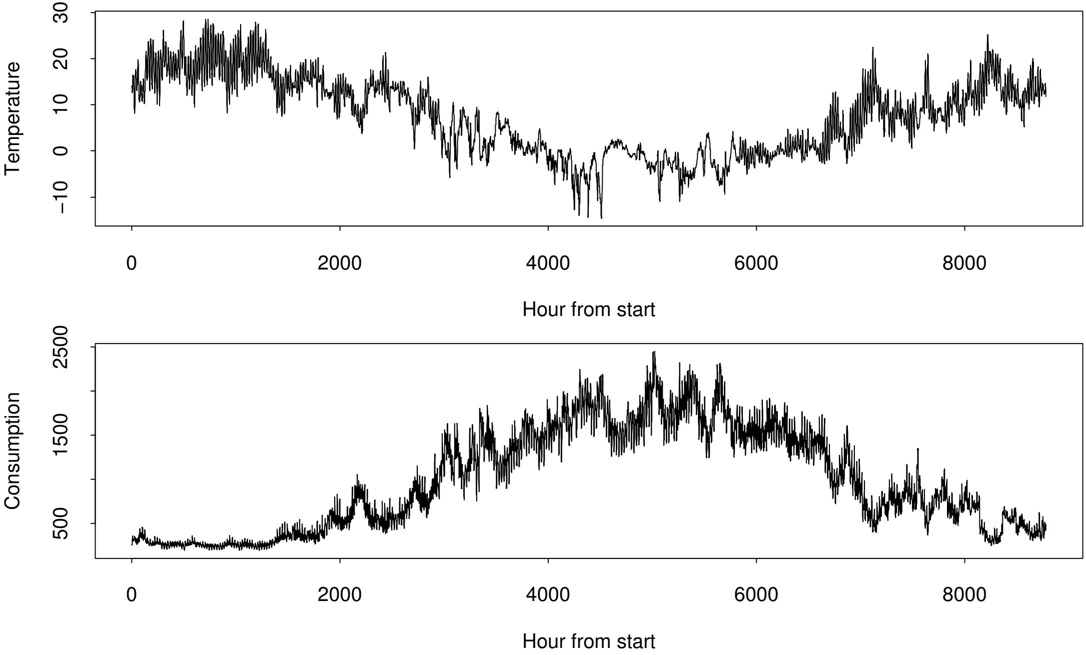
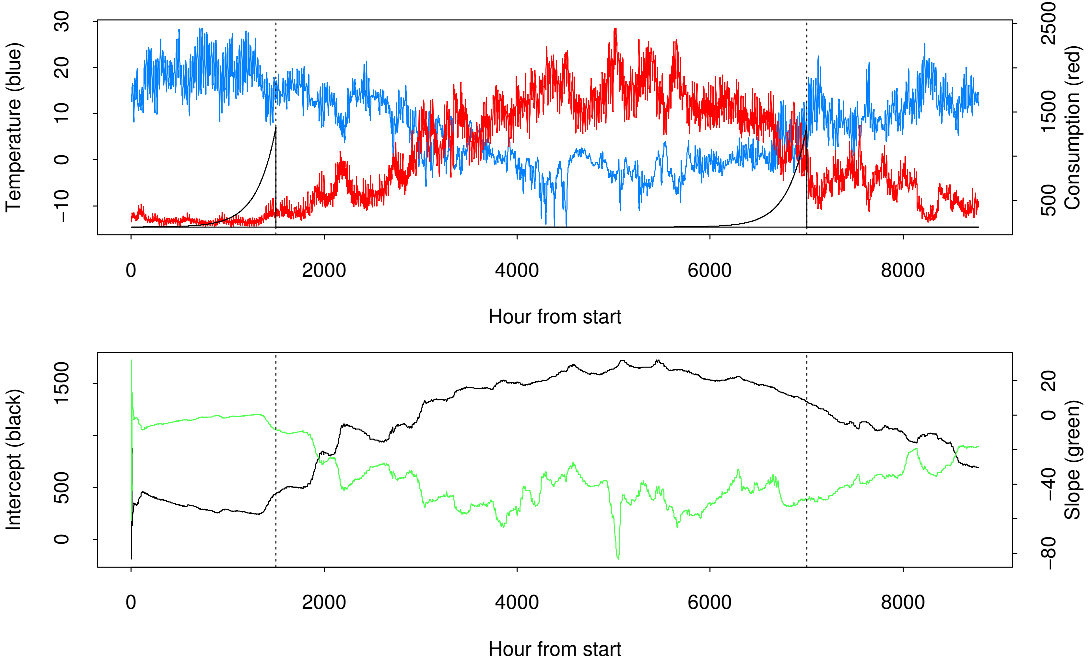
- What is the biggest danger in using variable forgetting? and how to solve it?
- Apply a lower bound λ_{min} on $\lambda(t)$
- Finally, try different values of S_0 to find one that results in an acceptable rate of updating parameter estimates.

Example: $HC_t = \mu + \theta_1 T_t + \varepsilon_t$



Example: $HC_t = \mu + \theta_1 T_t + \varepsilon_t$, $\lambda = 0.995$



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- How to circumvent this problem?
- Given the parameters, $\boldsymbol{\theta}$, the one-step prediction and thereby residuals can be calculated and used for regression.
- The model becomes

$$\widehat{Y}_{t|t-1}(\boldsymbol{\theta}) = \boldsymbol{X}_t^T(\boldsymbol{\theta})\boldsymbol{\theta},$$

where $\boldsymbol{X}_t^T(\boldsymbol{\theta})$ contains columns of residuals, assuming that the first few ones are zero.

Recursive pseudo-linear regression

- Problem: The ARMA structure cannot be estimated using regression directly. why?
- How to circumvent this problem?
- Given the parameters, $\boldsymbol{\theta}$, the one-step prediction and thereby residuals can be calculated and used for regression.
- The model becomes

$$\widehat{Y}_{t|t-1}(\boldsymbol{\theta}) = \boldsymbol{X}_t^T(\boldsymbol{\theta})\boldsymbol{\theta},$$

  where $\boldsymbol{X}_t^T(\boldsymbol{\theta})$ contains columns of residuals, assuming that the first few ones are zero.
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$$S_t(\boldsymbol{\theta}) = \lambda(t)S_{t-1}(\boldsymbol{\theta}) + (Y_t - \boldsymbol{X}_t^T(\boldsymbol{\theta})\theta)^2$$

  with respect to $\boldsymbol{\theta}$.

Recursive pseudo-linear regression

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- Then, the RPLR algorithm is:

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\boldsymbol{R}_t &= \boldsymbol{x}_t\boldsymbol{x}_t^T + \lambda(t)\boldsymbol{R}_{t-1} \\
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\widehat{\boldsymbol{\theta}}_t &= \boldsymbol{R}_t^{-1}\boldsymbol{h}_t
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- I.e. as before except that $\boldsymbol{x}_t$ must be calculated at each step.

Model-based adaptive estimation

- What is this?

$$
\begin{aligned}
\boldsymbol{X}_{t+1} &= \boldsymbol{X}_t + \boldsymbol{e}_{1,t}, \quad V(\boldsymbol{e}_{1,t}) = \boldsymbol{\Sigma}_1 \\
Y_t &= \boldsymbol{C}_t \boldsymbol{X}_t + \boldsymbol{e}_{2,t}, \quad V(\boldsymbol{e}_{2,t}) = \boldsymbol{\Sigma}_2
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- The parameters might even follow a linear model:

$$
\begin{aligned}
\boldsymbol{\theta}_{t+1} &= \boldsymbol{A}\boldsymbol{\theta}_t + \boldsymbol{e}_{1,t}, \quad V(\boldsymbol{e}_{1,t}) = \boldsymbol{\Sigma}_1 \\
Y_t &= \boldsymbol{X}_t^T \boldsymbol{\theta}_t + \boldsymbol{e}_{2,t}, \quad V(\boldsymbol{e}_{2,t}) = \boldsymbol{\Sigma}_2
\end{aligned}
$$