Time Series Analysis

Week 4 - Introduction to Stochastic Processes, Operators and Linear Systems

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Week 4: Outline of the lecture

- Operators; the backward shift operator; sec. 4.5.
- Stochastic processes in general: Sec 5.1, 5.2, 5.3 [except 5.3.2].

Operators; The backwards shift operator B

- An operator A is (here) a function of a time series $\{x_t\}$ (or a stochastic process $\{X_t\}$).
- Application of an operator on a time series $\{x_t\}$ yields a new time series $\{Ax_t\}$. Likewise of a stochastic process $\{AX_t\}$.
- Most important operator for us: The backwards shift operator $B : Bx_t = x_{t-1}$. Notice $B^j x_t = x_{t-j}$.
- All other operators we shall consider in this lecture may be expressed in terms of B.

The forward shift F

The forward shift operator

- $Fx_t = x_{t+1}$; $F^j x_t = x_{t+j}$; How can F be expressed in terms of B? ($Bx_t = x_{t-1}$)
- Combining a forward and backward shift yields the identity operator; $BFx_t = Bx_{t+1} = x_t$, ie. F and B are each others inverse: $B^{-1} = F$ and $F^{-1} = B$.

The difference operator ∇

The difference operator

- $\nabla x_t = x_t - x_{t-1} \ = (1 - B)x_t$.
- Thus: $\nabla = 1 - B$.

The summation S

$$
\begin{aligned}
Sx_t &= x_t + x_{t-1} + x_{t-2} + \ldots \\
&= x_t + Bx_t + B^2 x_t \ldots \\
&= (1 + B + B^2 + \ldots)x_t
\end{aligned}
$$

- Summation, then difference (using $Sx_t = x_t + Sx_{t-1}$)
$$\nabla Sx_t = Sx_t - Sx_{t-1} = x_t + Sx_{t-1} - Sx_{t-1} = x_t$$
- Difference, then summation
$$
\begin{aligned}
S\nabla x_t &= (1 + B + B^2 \ldots)x_t - (1 + B + B^2 \ldots)x_{t-1} \\
&= (1 + B + B^2 \ldots)x_t - (B + B^2 \ldots)x_t = x_t
\end{aligned}
$$
- So ∇ and S are each others inverse:
$$\nabla^{-1} = \frac{1}{1-B} = 1 + B + B^2 + \ldots = S$$

In R:

```r
x <- 1:10
x
diff(x)
cumsum(x)
# Not so nice in the ends!
cumsum(diff(x))
diff(cumsum(x))
# Put a zero in beginning to fix it
cumsum(diff(c(0,x)))
diff(cumsum(c(0,x)))
```

Shift operator is simply the most important:

```
lagvec <- function(x, lag){
    if (lag > 0) {
        ## Lag x, i.e. delay x lag steps
        return(c(rep(NA, lag), x[1:(length(x) - lag)]))
    }else if(lag < 0) {
        ## Lag x, i.e. delay x lag steps
        return(c(x[(abs(lag) + 1):length(x)], rep(NA, abs(lag))))
    }else{
        ## lag = 0, return x
        return(x)
    }
}
```

Lag to a data frame:

```
lagdf <- function(x, lag) {
  # Lag x, i.e. delay x lag steps
  if (lag > 0) {
    x[(lag+1):nrow(x), ] <- x[1:(nrow(x)-lag), ]
    x[1:lag, ] <- NA
  }else if(lag < 0) {
    # Lag x "ahead in time"
    x[1:(nrow(x)-abs(lag)), ] <- x[(abs(lag)+1):nrow(x), ]
    x[(nrow(x)-abs(lag)+1):nrow(x), ] <- NA
  }
  return(x)
}
```

Properties of B, F, ∇ and S

- The operators are all linear, ie.

$$H[ax_t + by_t] = aH[x_t] + bH[y_t]$$

- The operators may be combined into new operators:
  The power series

$$a(z) = \sum_{i=0}^{\infty} a_i z^i$$

defines a new operator from an operator H by linear combinations:

$$a(H) = \sum_{i=0}^{\infty} a_i H^i$$

Examples of combined operators

- ∇^{-1}:

$$\frac{1}{1-z} = \sum_{i=0}^{\infty} z^i \quad \text{so} \quad \nabla^{-1} = \frac{1}{1-B} = \sum_{i=0}^{\infty} B^i = S$$

- Operator polynomial of order q:

$$\theta(z) = \sum_{i=0}^{q} \theta_i z^i$$

ie. $\theta_i = 0$ for $i > q$.

$$\theta(B) = (1 + \theta_1 B + \cdots + \theta_q B^q)$$

where θ_0 is chosen to be 1

Stochastic Processes – in general

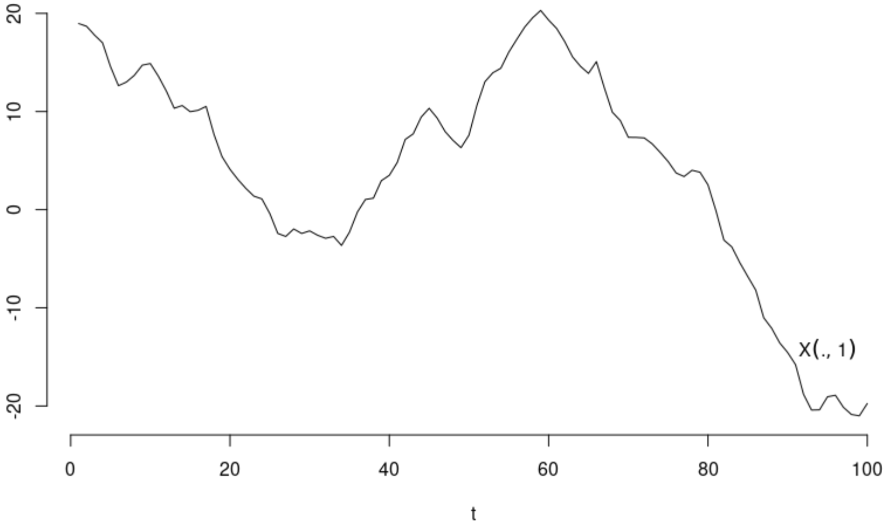
- Function: $X(t, \omega)$
- Time: $t \in T$
- Realization: $\omega \in \Omega$

- Index set: T
- Sample Space: Ω

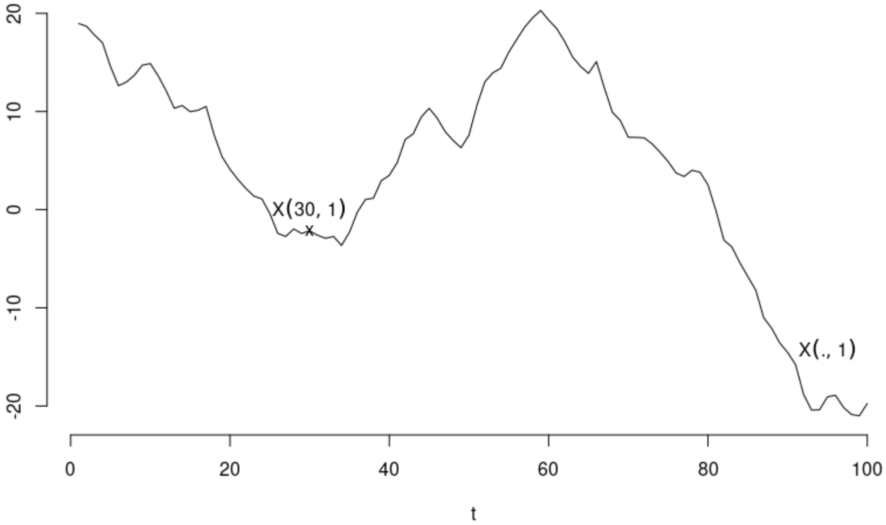
- $X(\cdot, \cdot)$ is a stochastic process
- $X(t, \cdot)$ is a random variable
- $X(\cdot, \omega)$ is a time series (i.e. *one* realization). This is what we often denote $\{x_t\}$.
- $X(t, \omega)$ is an observation. This is what we often denote x_t.

- In this course we restrict ourselves to the case where time is discrete, and the realizations take values on the real numbers (continuous range).

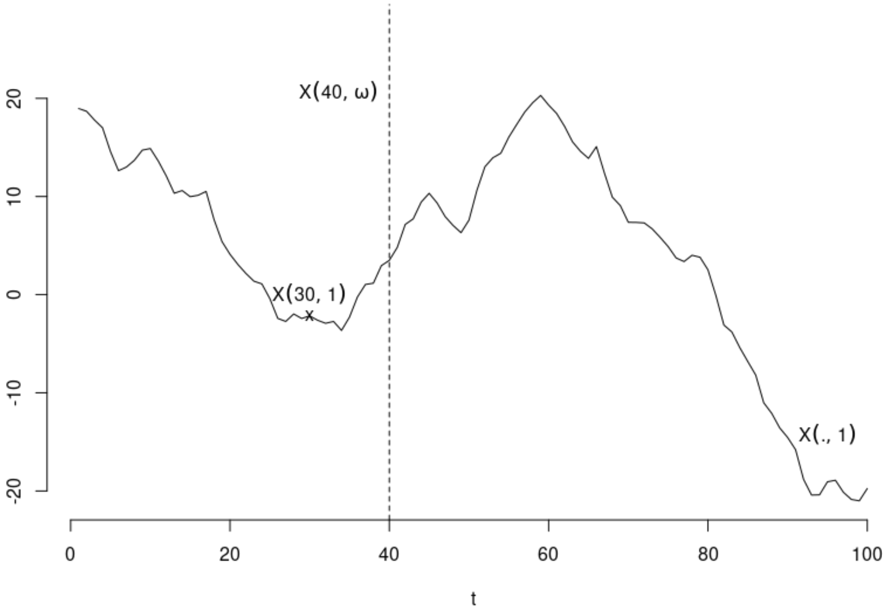
Stochastic Processes – illustration



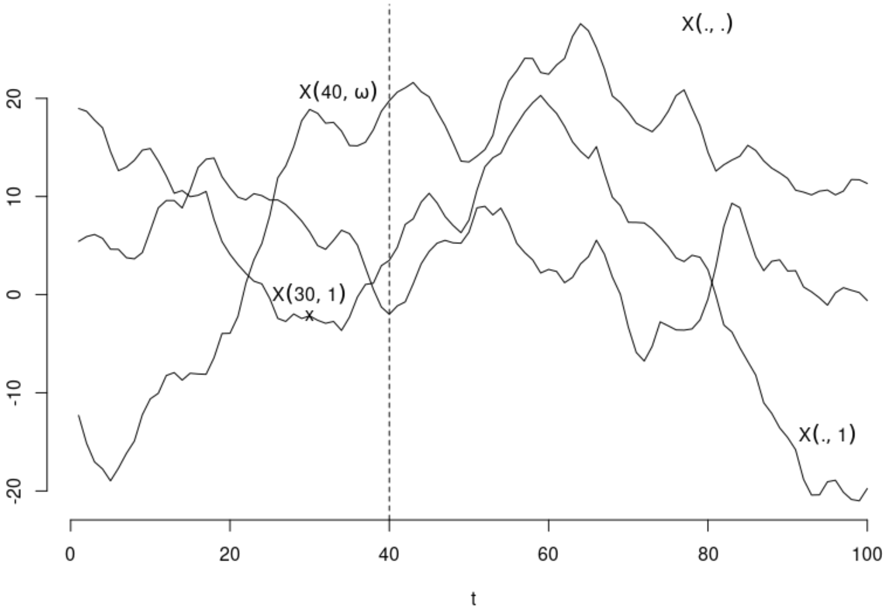
Stochastic Processes – illustration



Stochastic Processes – illustration



Stochastic Processes – illustration



Complete Characterization

In the end a stochastic process is just a multivariate variable. Thus, from lecture 1, it is characterised by its n-dimensional probability density:

$$f_{X(t_1),\ldots,X(t_n)}(x_1,\ldots,x_n)$$

2nd order moment representation

Mean function:

$$\mu(t) = E[X(t)] = \int_{-\infty}^{\infty} x \, f_{X(t)}(x) \, dx,$$

Autocovariance function:

$$\begin{aligned} \gamma_{XX}(t_1, t_2) &= \gamma(t_1, t_2) = \text{Cov}\left[X(t_1), X(t_2)\right] \\ &= E\left[(X(t_1) - \mu(t_1))(X(t_2) - \mu(t_2))\right] \end{aligned}$$

The variance function is obtained from $\gamma(t_1, t_2)$ when $t_1 = t_2 = t$:

$$\sigma^2(t) = V[X(t)] = E\left[(X(t) - \mu(t))^2\right]$$

Stationarity

- A process $\{X(t)\}$ is said to be *strongly stationary* if all finite-dimensional distributions are invariant for changes in time, i.e. for every n, and for any set $(t_1, t_2, \dots, t_n)$ and for any h it holds

$$f_{X(t_1),\cdots,X(t_n)}(x_1, \cdots, x_n) = f_{X(t_1+h),\cdots,X(t_n+h)}(x_1, \cdots, x_n)$$

- A process $\{X(t)\}$ is said to be *weakly stationary of order* k if all the first k moments are invariant to changes in time
- A weakly stationary process of order 2 is simply called *weakly stationary* or just *stationary*:

$$\mu(t) = \mu \quad \sigma^2(t) = \sigma^2 \quad \gamma(t_1, t_2) = \gamma(t_1 - t_2)$$

Ergodicity

- In time series analysis we normally assume that we have access to one realization only
- We therefore need to be able to determine characteristics of the process X_t from one realization x_t
- It is often enough to require the process to be mean-ergodic:

$$E[X(t)] = \int_{\Omega} x(t, \omega) f(\omega)\, d\omega = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t, \omega)\, dt$$

i.e. if the *mean of the ensemble* equals the *mean over time*

Some intuitive examples, not directly related to time series:
`http://news.softpedia.com/news/What-is-ergodicity-15686.shtml`

Special processes

- *Normal processes* (also called *Gaussian processes*): All finite-dimensional distribution functions are (multivariate) normal distributions
- *Markov processes*: The conditional distribution depends only on the latest state of the process:

$$\mathrm{P}\{X(t_n) \leq x | X(t_{n-1}), \cdots, X(t_1)\} = \mathrm{P}\{X(t_n) \leq x | X(t_{n-1})\}$$

- *Deterministic processes*: Can be predicted without uncertainty from past observations
- *Pure stochastic processes*: Can be written as a linear combination of uncorrelated random variables
- Decomposition: $X_t = S_t + D_t$, where S_t is a pure stochastic process and D_t is a deterministic process.

Autocovariance and autocorrelation

- For stationary processes: Only dependent on the time difference $\tau = t_2 - t_1$
- Autocovariance:

$$\gamma(\tau) = \gamma_{XX}(\tau) = \mathrm{Cov}[X(t), X(t+\tau)] = E[X(t)X(t+\tau)]$$

- Autocorrelation:

$$\rho(\tau) = \rho_{XX}(\tau) = \gamma_{XX}(\tau)/\gamma_{XX}(0) = \gamma_{XX}(\tau)/\sigma_X^2$$

- Some properties of the autocovariance function:
  - $\gamma(\tau) = \gamma(-\tau)$
  - $|\gamma(\tau)| \leq \gamma(0)$

White noise

- Def. 5.9: $\{\varepsilon_t\}$ is a sequence of mutually uncorrelated identically distributed random variables, where:
  - $\mu_t = \text{E}[\varepsilon_t] = 0$
  - $\sigma_t = \text{Var}[\varepsilon_t] = \sigma_\varepsilon^2$
  - $\gamma_\varepsilon(k) = \text{Cov}[\varepsilon_t, \varepsilon_{t+k}] = 0$, for $k \neq 0$

Linear processes

- A linear process $\{Y_t\}$ is a process that can be written on the form

$$Y_t - \mu = \sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}$$

  where μ is the mean value of the process and
- $\{\varepsilon_t\}$ is white noise, i.e. a sequence of uncorrelated, identically distributed random variables.
- $\{\varepsilon_t\}$ can be scaled so that $\psi_0 = 1$
- Without loss of generality we assume $\mu = 0$

ψ- and π-weights

- Transfer function and linear process:

$$\psi(B) = 1 + \sum_{i=1}^{\infty} \psi_i B^i \qquad Y_t = \psi(B)\varepsilon_t$$

- Inverse operator (if it exists) and the linear process:

$$\pi(B) = 1 + \sum_{i=1}^{\infty} \pi_i B^i \qquad \pi(B) Y_t = \varepsilon_t,$$

- Autocovariance using ψ-weights:

$$\gamma(k) = \mathrm{Cov}\left[Y_t, Y_{t+k}\right] = \mathrm{Cov}\left[\sum_{i=0}^{\infty} \psi_i \varepsilon_{t-i}, \sum_{i=0}^{\infty} \psi_i \varepsilon_{t+k-i}\right] = \sigma_\varepsilon^2 \sum_{i=0}^{\infty} \psi_i \psi_{i+k}$$

Stationarity and invertibility

- The linear process $Y_t = \psi(B)\varepsilon_t$ is *stationary* if

$$\psi(z) = \sum_{i=0}^{\infty} \psi_i z^{-i}$$

converges for $|z| \geq 1$ (i.e. old values of ε_t are down-weighted)

- The linear process $\pi(B)Y_t = \varepsilon_t$ is said to be *invertible* if

$$\pi(z) = \sum_{i=0}^{\infty} \pi_i z^{-i}$$

converges for $|z| \geq 1$ (i.e. ε_t can be calculated from recent values of Y_t)

Linear process as a statistical model?

$$Y_t = -\pi_1 Y_{t-1} - \pi_2 Y_{t-2} \ldots + \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + +\psi_3 \varepsilon_{t-3} + \ldots$$

- Observations: Y_1, Y_2, $Y_3, \ldots,$ Y_N
- Task: Find an infinite number of parameters from N observations!
- Solution: Restrict the sequence $1, \psi_1, \psi_2, \psi_3, \ldots$ - Which gives us the famous ARMA model.