Time Series Analysis

Week 5 - AR, MA and ARMA processes

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Week 5: Outline of the lecture

- Stochastic processes - 2nd part:
    - MA, AR, and ARMA-processes, Sec. 5.5
    - Non-stationary models, Sec. 5.6
    - Seasonal ARIMA models
    - Optimal Prediction, Sec. 5.7
- Estimation of parameters in linear dynamic models, Sec. 6.4

Linear process as a statistical model?

$$Y_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \psi_3 \varepsilon_{t-3} + \ldots$$

- ▶ Observations: $Y_1, Y_2, Y_3, \ldots, Y_N$
- ▶ Task: Find an infinite number of parameters from N observations!
- ▶ Solution:

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- ▶ Observations: $Y_1, Y_2, Y_3, \ldots, Y_N$
- ▶ Task: Find an infinite number of parameters from N observations!
- ▶ Solution: Restrict the sequence $1, \psi_1, \psi_2, \psi_3, \ldots$

MA(q), AR(p), and ARMA(p, q) processes

$$Y_t = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \ldots + \theta_q \varepsilon_{t-q}$$

$\{\varepsilon_t\}$ is white noise

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$$Y_t = \theta(B)\varepsilon_t$$
$$\phi(B)\,Y_t = \varepsilon_t$$
$$\phi(B)\,Y_t = \theta(B)\varepsilon_t$$

where

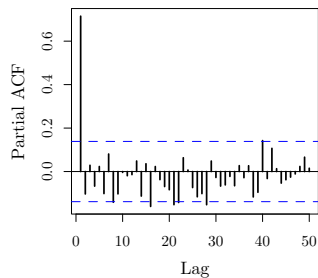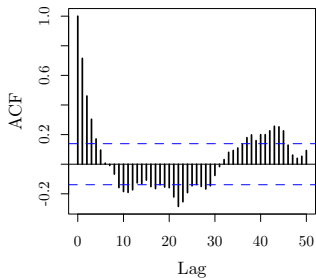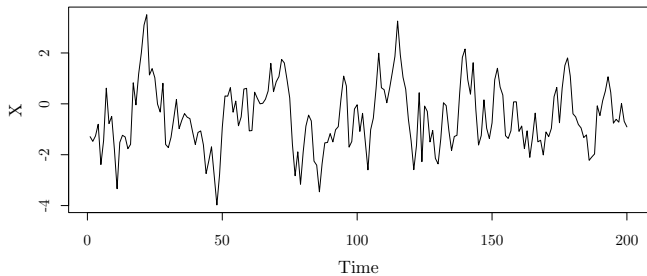$$\phi(B) = (1 + \phi_1 B + \phi_1 B^2 + \ldots + \phi_p B^p)$$
$$\theta(B) = (1 + \theta_1 B + \theta_2 B^2 + \ldots + \phi_q B^q)$$

are polynomials in the backward shift operator B, ($BX_t = X_{t-1}$, $B^2 X_t = X_{t-2}$)
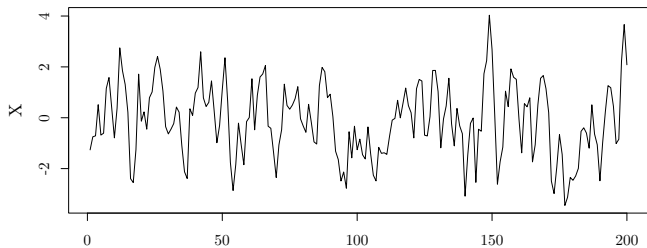
```
# Write an ARMA simulation
n <- 20000
x <- numeric(n)
# AR1:
# (1 - phi1 * B) * X_t = eps_t
#
```

```r
# Function for plot
plotit <- function(x){
  layout(rbind(1,2:3))
  par(mar=c(3,3,1,1), mgp=c(2, 0.7,0))
  plot(x, ylab="X")
  acf(x, lag.max=50, lwd=2)
  pacf(x, lag.max=50, lwd=2)
}
```
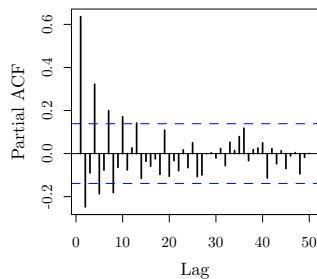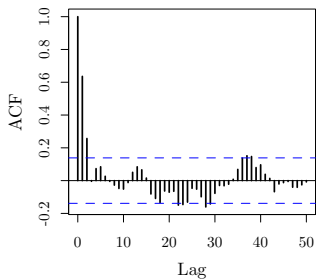
```
# AR(1)
n <- 200
plotit( arima.sim(model=list(ar=c(0.8)), n=n) )
```
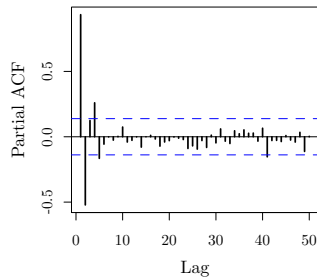
```
# MA(2)
plotit( arima.sim(list(ma=c(0.9, 0.8)), n) )
```

```
# arma(1,2)
plotit( arima.sim(list(ar=c(0.8),ma=c(0.9, 0.8)), n) )
```
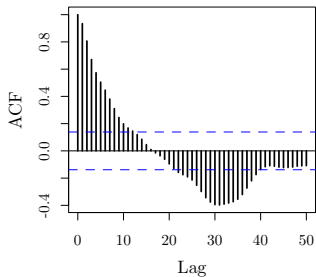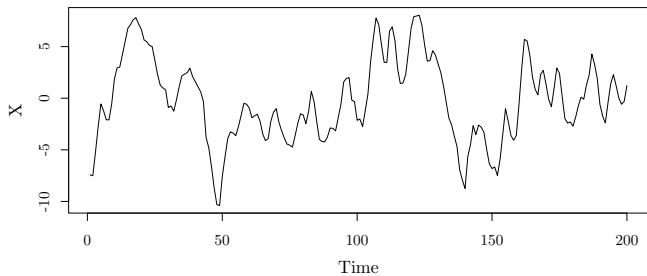
```r
# A simulation function for ARMA simulation, use model as arima.sim, i.e. flip sign of phi (into ar) co-
efficients
sim <- function(model, n, nburnin=100){
  n <- n + nburnin
  # Take the ar and ma part
  ar <- model$ar
  ma <- model$ma
  # The order (i.e. the number of lags)
  p <- length(ar)
  q <- length(ma)
  # The vector for the simulation result
  y <- numeric(n)
  # Generate the random normal values
  eps <- rnorm(n)
  # Run the simulation
  for(i in (max(p,q)+1):n){
    y[i] <- eps[i] + sum(y[i-(1:p)] * ar) + sum(eps[i-(1:q)] * ma)
  }
  # Return without the burn-in period
  return(y[(nburnin+1):n])
}
```

```r
# Test it by comparing
model <- list(ar=c(0.4), ma=c(0.2))
set.seed(12)
sim(model, 10, nburnin=100)
set.seed(12)
arima.sim(model, 10, n.start=100)

# Non-stationary process
# Do the simulation and plot
n <- 200
model <- list(ar=c(1.01))
#arima.sim(model, n)
x <- sim(model, n)
plot(x, type="l", ylab="x", xlab="t")
```
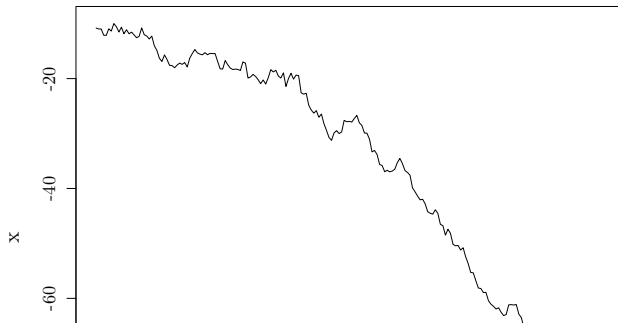
Invertibility and Stationarity

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- A stochastic process is said to be *stationary* if its distribution does not change over time.

Invertibility and Stationarity of ARMA models

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- $ARMA(p, q)$
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  - Invertible if the roots in $\theta(z^{-1})$ with respect to z all are within the unit circle

Autocorrelations

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$$Y_t = (1 + 0.9B + 0.8B^2)\varepsilon_t$$

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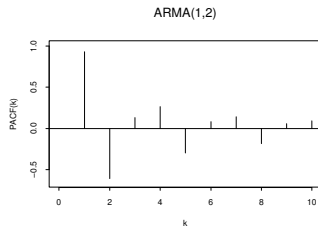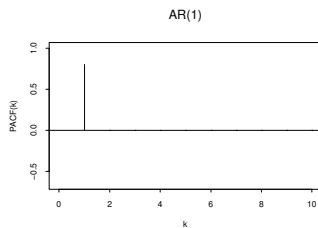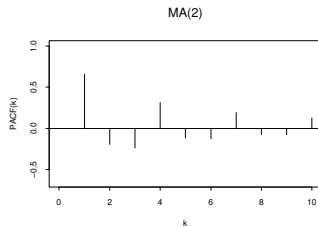
Non-stationary time series
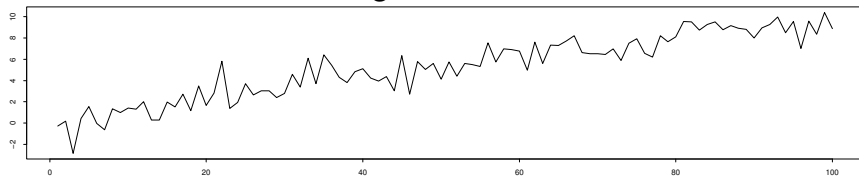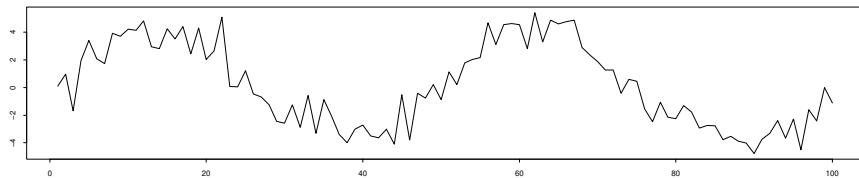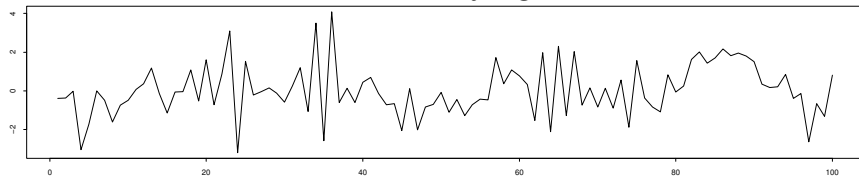
Differencing
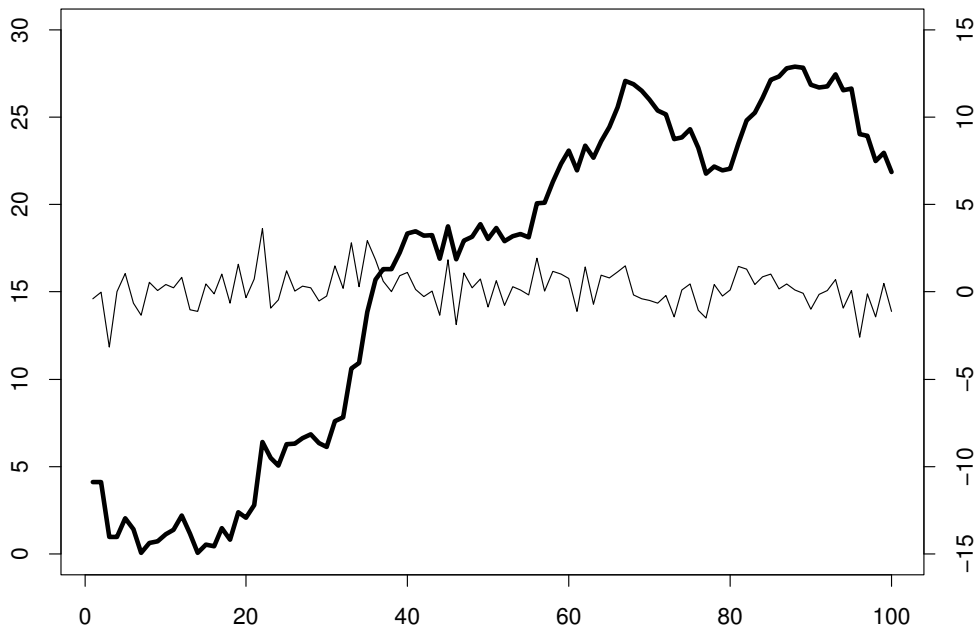
The $ARIMA(p, d, q)$-process

▶ An $ARMA(p, q)$ model for:
$$W_t = \nabla^d Y_t = (1 - B)^d Y_t$$

where $\{Y_t\}$ is the series

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▶ If we consider stationarity:
$$\phi(z^{-1})(1 - z^{-1})^d = 0$$

i.e. d roots in $z = 1 + 0i$, and the rest inside the unit circle

Seasonal Models

- In general, would you rather use new or old information in your models, for example would you prefer $Y_t = \theta Y_{t-1} + \epsilon_t$ or $Y_t = \theta Y_{t-2} + \epsilon_t$?

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- In general, would you rather use new or old information in your models, for example would you prefer $Y_t = \theta Y_{t-1} + \epsilon_t$ or $Y_t = \theta Y_{t-2} + \epsilon_t$?
- When and why would it make sense to prefer older information over newer information?

The $(p, d, q) \times (P, D, Q)_s$ seasonal process

- A multiplicative (stationary) $ARMA(p, q)$ model for:

$$W_t = \nabla^d \nabla_s^D \, Y_t = (1 - B)^d (1 - B^s)^D \, Y_t$$

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- If we consider stationarity:

$$\phi(z^{-1})\Phi(z^{-s})(1 - z^{-1})^d (1 - z^{-s})^D = 0$$

  i.e. d roots in $z = 1 + 0i$, $D \times s$ roots on the unit circle, and the rest inside the unit circle

The case $d = D = 0$; stationary seasonal process

▶ General:
$$\phi(B)\Phi(B^s)\,Y_t = \theta(B)\Theta(B^s)\varepsilon_t$$

▶ Example:
$$(1 - \Phi B^{12})\,Y_t = \varepsilon_t$$

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▶ Which can also be written:
$$Y_t = \Phi\,Y_{t-12} + \varepsilon_t$$

i.e. Y_t depend on $Y_{t-12}, \ Y_{t-24}, \ \ldots$ (thereof the name)

▶ How would you think that the auto correlation function looks?

ACF and PACF of seasonal ARMA models

ACF
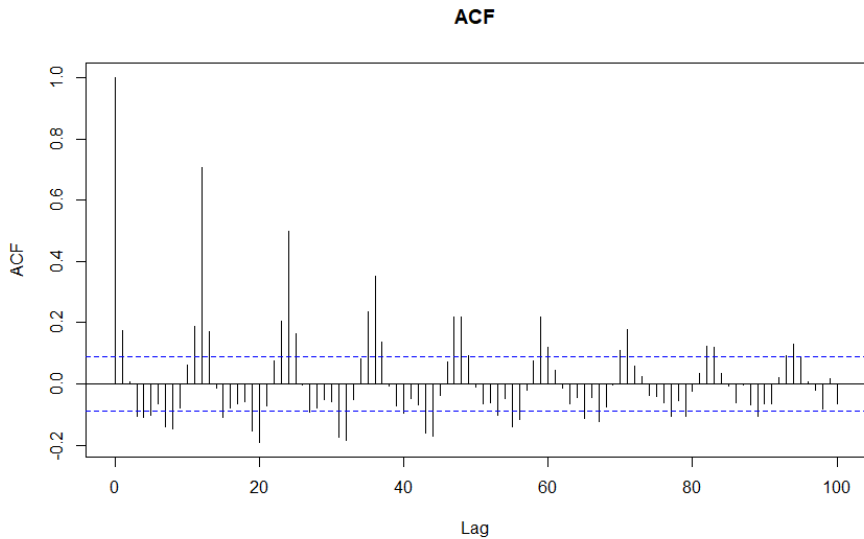
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PACF
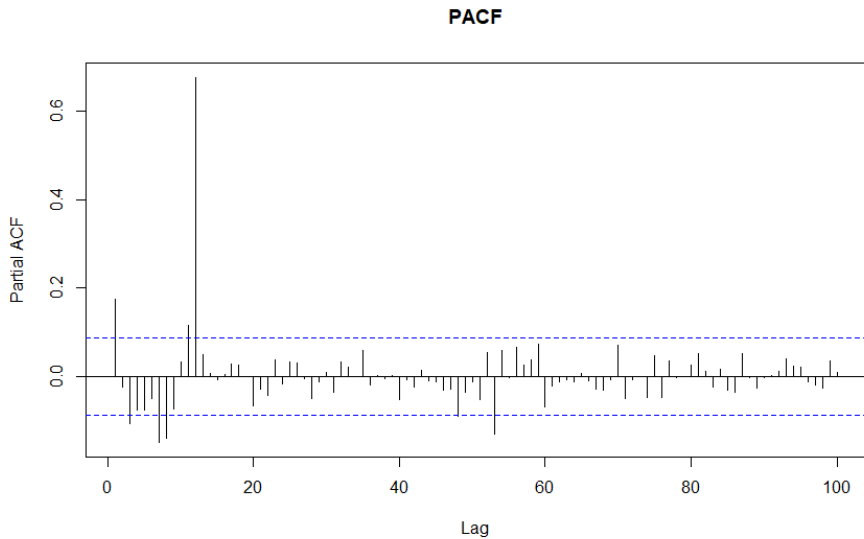
Prediction

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- We want a prediction of Y_{t+k}, where $k \geq 1$

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- We want a prediction of Y_{t+k}, where $k \geq 1$
- Thus, we want the conditional expectation:

$$\widehat{Y}_{t+k|t} = E[Y_{t+k}|Y_t, Y_{t-1}, Y_{t-2}, \ldots]$$

Example – prediction in the $AR(1)$ model

- We write the model like $Y_{t+1} = \phi Y_t + \varepsilon_{t+1}$ (note the sign on ϕ)
- 1-step prediction:

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▶ 2-step prediction:

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&= E[\phi Y_t + \varepsilon_{t+1}|Y_t, Y_{t-1}, \ldots] \\
&= \phi Y_t + 0 = \phi Y_t
\end{aligned}
$$

▶ 2-step prediction:

$$
\begin{aligned}
\widehat{Y}_{t+2|t} &= E[Y_{t+2}|Y_t, Y_{t-1}, \ldots] \\
&= E[\phi Y_{t+1} + \varepsilon_{t+2}|Y_t, Y_{t-1}, \ldots] \\
&= \phi \widehat{Y}_{t+1|t} + 0 \\
&= \phi^2 Y_t
\end{aligned}
$$

Example – prediction in the $AR(1)$ model

- We write the model like $Y_{t+1} = \phi Y_t + \varepsilon_{t+1}$ (note the sign on ϕ)
- 1-step prediction:

$$
\begin{aligned}
\widehat{Y}_{t+1|t} &= E[Y_{t+1}|Y_t, Y_{t-1}, \ldots] \\
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&= \phi \widehat{Y}_{t+1|t} + 0 \\
&= \phi^2 Y_t
\end{aligned}
$$

- k-step prediction: $\boxed{\widehat{Y}_{t+k|t} = \phi^k Y_t}$

Example – prediction in $Y_t = 0.8\,Y_{t-1} + \varepsilon_t$
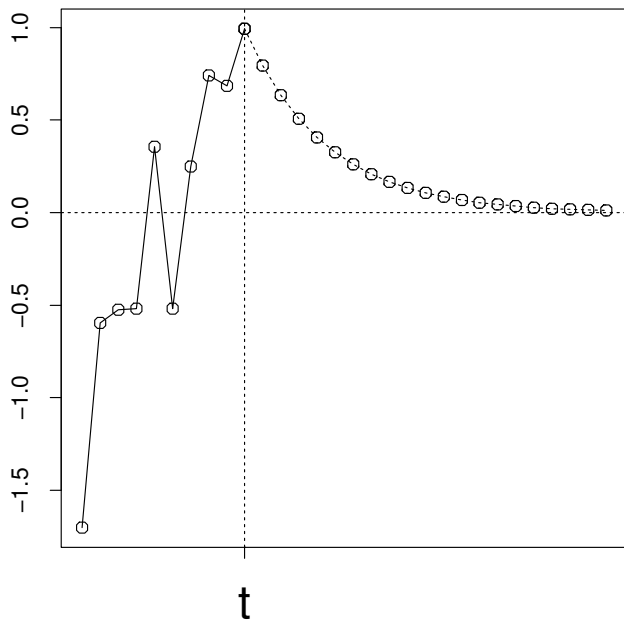
Variance of prediction error for the $AR(1)$-process

Prediction error:

$$e_{t+k|t} = Y_{t+k} - \widehat{Y}_{t+k|t}$$

Variance of prediction error for the $AR(1)$-process

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Bring it on psi-form (MA-form):

$$Y_{t+k} = \phi Y_{t+k-1} + \varepsilon_{t+k}$$

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Bring it on psi-form (MA-form):

$$\begin{aligned} Y_{t+k} &= \phi Y_{t+k-1} + \varepsilon_{t+k} \\ &= \phi(\phi Y_{t+k-2} + \varepsilon_{t+k-1}) + \varepsilon_{t+k} \end{aligned}$$

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\end{aligned}
$$

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&= \phi^2 Y_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^2(\phi Y_{t+k-3} + \varepsilon_{t+k-2}) + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}
\end{aligned}
$$

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&= \phi^2(\phi Y_{t+k-3} + \varepsilon_{t+k-2}) + \phi \varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&= \phi^3 Y_{t+k-3} + \phi^2 \varepsilon_{t+k-2} + \phi \varepsilon_{t+k-1} + \varepsilon_{t+k}
\end{aligned}$$

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&= \phi^3 Y_{t+k-3} + \phi^2\varepsilon_{t+k-2} + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k} \\
&\vdots \\
&= \phi^k Y_t + \phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}
\end{aligned}
$$

Variance of prediction error for the $AR(1)$-process

Variance of prediction error:

$$V[e_{t+k|t}] \;=\; V[\phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}]$$

Variance of prediction error for the $AR(1)$-process

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V[e_{t+k|t}] &= V[\phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}] \\
&= (\phi^{2(k-1)} + \phi^{2(k-2)} + \ldots + \phi^2 + 1)\sigma_\varepsilon^2
\end{aligned}
$$

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V[e_{t+k|t}] &= V[\phi^{k-1}\varepsilon_{t+1} + \phi^{k-2}\varepsilon_{t+2} + \ldots + \phi\varepsilon_{t+k-1} + \varepsilon_{t+k}] \\
&= (\phi^{2(k-1)} + \phi^{2(k-2)} + \ldots + \phi^2 + 1)\sigma_\varepsilon^2
\end{aligned}
$$

$(1 - \alpha) \times 100\%$ prediction interval:

$$
\widehat{Y}_{t+k|t} \pm u_{\alpha/2}\sqrt{V[e_{t+k|t}]}
$$

$u_{\alpha/2}$ is the $\alpha/2$-quantile in the standard normal distribution

Estimation

- Assume that we have an appropriate model structure $AR(p)$, $MA(q)$, $ARMA(p, q)$, $ARIMA(p, d, q)$ with p, d, and q known
- **Task**: Based on the observations find appropriate values of the parameters

Estimation

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- **Task**: Based on the observations find appropriate values of the parameters
- The book describes many methods:
    - Moment estimates
    - LS-estimates
    - Prediction error estimates
        - Conditioned
        - Unconditioned
    - ML-estimates
        - Conditioned
        - Unconditioned (exact)

Estimation in AR(2) model

- Observations: $y_1, \ y_2, \ \ldots, \ y_N$

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$$
\begin{aligned}
y_3 &= \phi_1 y_2 + \phi_2 y_1 + e_3 \\
y_4 &= \phi_1 y_3 + \phi_2 y_2 + e_4 \\
y_5 &= \phi_1 y_4 + \phi_2 y_3 + e_5 \\
&\vdots \\
y_N &= \phi_1 y_{N-1} + \phi_2 y_{N-2} + e_N
\end{aligned}
$$

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\end{aligned}
$$

$$
\begin{bmatrix} y_3 \\ \vdots \\ y_N \end{bmatrix}
=
\begin{bmatrix} -y_2 & -y_1 \\ \vdots & \vdots \\ -y_{N-1} & -y_{N-2} \end{bmatrix}
\begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}
+
\begin{bmatrix} e_{3|2} \\ \vdots \\ e_{N|N-1} \end{bmatrix}
$$

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$$

$$
\begin{bmatrix} y_3 \\ \vdots \\ y_N \end{bmatrix} = \begin{bmatrix} -y_2 & -y_1 \\ \vdots & \vdots \\ -y_{N-1} & -y_{N-2} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix} + \begin{bmatrix} e_{3|2} \\ \vdots \\ e_{N|N-1} \end{bmatrix}
$$

Or just:

$$
\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\theta} + \boldsymbol{\varepsilon}
$$

Solution

To minimize the sum of the squared 1-step prediction errors $\boldsymbol{\epsilon}^T \boldsymbol{\epsilon}$ we use the result for the General Linear Model from Chapter 3:

$$\widehat{\boldsymbol{\theta}} = (\boldsymbol{X}^T \boldsymbol{X})^{-1} \boldsymbol{X}^T \boldsymbol{Y}$$

With

$$\boldsymbol{X} = \begin{bmatrix} -y_2 & -y_1 \\ \vdots & \vdots \\ -y_{N-1} & -y_{N-2} \end{bmatrix} \quad \text{and} \quad \boldsymbol{Y} = \begin{bmatrix} y_3 \\ \vdots \\ y_N \end{bmatrix}$$

▶ Asymptotically: $V(\hat{\theta}) = \sigma_\epsilon^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}$

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▶ Asymptotically: $V(\hat{\theta}) = \sigma_\epsilon^2 (\boldsymbol{X}^T \boldsymbol{X})^{-1}$
▶ How does it generalize to AR(p)-models?
▶ How about ARMA(p,q)-models?

Least squares for AR

```
# Test it by comparing
model <- list(ar=c(0.4))
set.seed(12)
sim(model, 10, nburnin=100)
set.seed(12)
x <- arima.sim(model, 100)

X <- lagdf(x, 0:3)
summary(lm(k0 ~ k1, X))
summary(lm(k0 ~ k1 + k2, X))
summary(lm(k0 ~ k1 + k2 + k3, X))
```

Maximum Likelihood estimates

- $ARMA(p, q)$-process:

$$Y_t + \phi_1 Y_{t-1} + \ldots + \phi_p Y_{t-p} = \varepsilon_t + \theta_1 \varepsilon_{t-1} + \ldots + \theta_q \varepsilon_{t-q}$$

- Notation:

$$\boldsymbol{\theta}^T = (\phi_1, \ldots, \phi_p, \theta_1, \ldots, \theta_q)$$
$$\mathbf{Y}_t^T = (Y_t, Y_{t-1}, \ldots, Y_1)$$

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- The Likelihood function is the joint probability distribution function for all observations for given values of $\boldsymbol{\theta}$ and σ_ε^2:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = f(\mathbf{Y}_N | \boldsymbol{\theta}, \sigma_\varepsilon^2)$$

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- Given the observations $\mathbf{Y}_N$ we estimate $\boldsymbol{\theta}$ and σ_ε^2 as the values for which the likelihood is maximized.

Likelihood example

```
# see the week5_example_likelihood.R script
```

The likelihood function for $ARMA(p, q)$-models

▶ The random variable $Y_N | \mathbf{Y}_{N-1}$ only contains ε_N as a random component

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- The random variable $Y_N | \mathbf{Y}_{N-1}$ only contains ε_N as a random component
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$$f(\mathbf{Y}_N | \boldsymbol{\theta}, \sigma_\varepsilon^2) = f(Y_N | \mathbf{Y}_{N-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) f(\mathbf{Y}_{N-1} | \boldsymbol{\theta}, \sigma_\varepsilon^2)$$

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- Repeating these arguments:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = \left(\prod_{t=p+1}^{N} f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) \right) f(\mathbf{Y}_p|\boldsymbol{\theta}, \sigma_\varepsilon^2)$$

Evaluating the conditional likelihood function

▶ **Task**: Find the conditional 1-step densities, $f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2)$, given specified values of the parameters $\boldsymbol{\theta}$ and σ_ε^2

Evaluating the conditional likelihood function

- **Task**: Find the conditional 1-step densities, $f(Y_t | \mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2)$, given specified values of the parameters $\boldsymbol{\theta}$ and σ_ε^2
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- We assume that the process is Gaussian:

$$f(Y_t|\mathbf{Y}_{t-1}, \boldsymbol{\theta}, \sigma_\varepsilon^2) = \frac{1}{\sigma_\varepsilon\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2}(Y_t - \widehat{Y}_{t|t-1}(\boldsymbol{\theta}))^2\right)$$

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- And therefore:

$$L(\mathbf{Y}_N; \boldsymbol{\theta}, \sigma_\varepsilon^2) = (\sigma_\varepsilon^2 2\pi)^{-\frac{N-p}{2}} \exp\left(-\frac{1}{2\sigma_\varepsilon^2} \sum_{t=p+1}^{N} \varepsilon_t^2(\boldsymbol{\theta})\right)$$

ML-estimates

▶ The (conditional) ML-estimate $\widehat{\boldsymbol{\theta}}$ is a prediction error estimate
  since it is obtained by minimizing

$$S(\boldsymbol{\theta}) = \sum_{t=p+1}^{N} \varepsilon_t^2(\boldsymbol{\theta})$$

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▶ By differentiating w.r.t. σ_ε^2 it can be shown that the ML-estimate of σ_ε^2 is (remember that p is the order of the AR part):

$$\widehat{\sigma}_\varepsilon^2 = S(\widehat{\boldsymbol{\theta}})/(N-p)$$

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▶ The estimate $\widehat{\boldsymbol{\theta}}$ is asymptotically unbiased and efficient, and the variance-covariance matrix is approximately

$$2\sigma_\varepsilon^2 \boldsymbol{H}^{-1}$$

where $\boldsymbol{H}$ contains the 2nd order partial derivatives of $S(\boldsymbol{\theta})$ at the minimum

Finding the ML-estimates using the PE-method

- 1-step predictions:

$$\widehat{Y}_{t+1|t} = -\phi_1 Y_t - \ldots - \phi_p Y_{t-p+1} + \theta_1 \varepsilon_t + \ldots + \theta_q \varepsilon_{t-q+1}$$

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▶ If we use (Condition on) $\varepsilon_p = \varepsilon_{p-1} = \ldots = \varepsilon_{p+1-q} = 0$ we can find:

$$\widehat{Y}_{p+1|p} = -\phi_1 Y_p - \ldots - \phi_p Y_1 + \theta_1 \varepsilon_p + \ldots + \theta_q \varepsilon_{p-q+1}$$

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▶ Which will give us $\varepsilon_{p+1} = Y_{p+1} - \widehat{Y}_{p+1|p}$ and we can then calculate $\widehat{Y}_{p+2|p+1}$ and ε_{p+2}
  . . . and so on until we have all the 1-step prediction errors we need.

Finding the ML-estimates using the PE-method

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▶ We use numerical optimization to find the parameters which minimize the sum of squared prediction errors