Time Series Analysis

Solutions to problems in Chapter 10

IMM

Solution 10.1

Question 1.

The predictions are given - as always - by the conditional expectation

$$\begin{aligned}
\hat{\mathbf{X}}(t_0+1) &= E[\mathbf{X}(t_0+1)|\mathbf{X}(t_0),\mathbf{X}(t_0-1),\ldots] \\
&= E\left[\begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} X_i(t_0) \\ X_m(t_0) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} U_r(t_0) + \right. \\
&\quad \left. \begin{bmatrix} \epsilon_1(t_0+1) \\ \epsilon_2(t_0+1) \end{bmatrix} |\mathbf{X}(t_0),\mathbf{X}(t_0+1),\ldots\right] \\
&= \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} -3.0 \\ -1.2 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} 2 = \begin{bmatrix} -0.74 \\ -1.22 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{X}}(t_0+2) &= \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} \hat{X}_i(t_0+1) \\ \hat{X}_m(t_0+1) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} U_r(t_0+1) \\
&= \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} -0.74 \\ -1.22 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} 2 = \begin{bmatrix} 0.84 \\ -1.15 \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\hat{\mathbf{X}}(t_0+3) &= \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} \hat{X}_i(t_0+2) \\ \hat{X}_m(t_0+2) \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} U_r(t_0+2) \\
&= \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}\begin{bmatrix} 0.84 \\ -1.15 \end{bmatrix} + \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} 2 = \begin{bmatrix} 1.96 \\ 1.02 \end{bmatrix}
\end{aligned}$$

Since $U_r(t)$ is a deterministic input to the system, the variance contribution is only from ϵ_t and the variance of the ℓ-step prediction is given by Equation (10.102), where

$$\mathbf{A} = \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix}$$

This gives the following variances

$$\ell = 1: \quad \mathbf{\Sigma_1} = \mathbf{\Sigma} = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix}$$

$$\begin{aligned}\ell = 2: \quad \mathbf{\Sigma_2} &= \mathbf{\Sigma_1} + \mathbf{A}\mathbf{\Sigma_1}\mathbf{A}^\top \\ &= \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix} + \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix} \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix} \begin{bmatrix} 0.70 & 0.04 \\ 0.20 & 0.95 \end{bmatrix} \\ &= \begin{bmatrix} 0.060 & 0.003 \\ 0.003 & 0.019 \end{bmatrix}\end{aligned}$$

$$\begin{aligned}\ell = 3: \quad \mathbf{\Sigma_3} &= \mathbf{\Sigma_1} + \mathbf{A}\mathbf{\Sigma_2}\mathbf{A}^\top \\ &= \begin{bmatrix} 0.04 & 0 \\ 0 & 0.01 \end{bmatrix} + \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix} \begin{bmatrix} 0.060 & 0.003 \\ 0.003 & 0.019 \end{bmatrix} \begin{bmatrix} 0.70 & 0.04 \\ 0.20 & 0.95 \end{bmatrix} \\ &= \begin{bmatrix} 0.071 & 0.007 \\ 0.007 & 0.028 \end{bmatrix}\end{aligned}$$

Question 2.

At the new reference level holds

$$E[\mathbf{X}(t)] = \begin{bmatrix} E[X_i(t)] \\ E[X_m(t)] \end{bmatrix} = \begin{bmatrix} \mu_i \\ \mu_m \end{bmatrix} = \mu$$

Since the mean value of $\epsilon(t)$ is zero

$$\mu = \mathbf{A}\mu + \mathbf{B}U_r$$

where

$$\mathbf{A} = \begin{bmatrix} 0.70 & 0.20 \\ 0.04 & 0.95 \end{bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix}$$

Rewriting the above equation we get

$$\begin{aligned}\mu &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U_r \\ &= \begin{bmatrix} 1 - 0.70 & -0.20 \\ -0.04 & 1 - 0.95 \end{bmatrix}^{-1} \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} U_r = \begin{bmatrix} 7.14 & 28.6 \\ 5.71 & 42.9 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0.02 \end{bmatrix} U_r \\ &= \begin{bmatrix} 6.24 \\ 5.44 \end{bmatrix} U_r = \begin{bmatrix} 12.56 \\ 10.88 \end{bmatrix}\end{aligned}$$

where $U_r = 2$. The old reference temperature was 21°C, so the new reference level will be 33.56°C for the indoor temperature and 31.88°C for the wall temperature.

Solution 10.2

Question 1.

$$\begin{bmatrix} 1-0.8B & 0.6B \\ -0.4B & 1-0.7B \end{bmatrix} \begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1-0.4B \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} \Leftrightarrow \tag{1}$$
$$\begin{bmatrix} X_{1,t} \\ X_{2,t} \end{bmatrix} - \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} = \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} \epsilon_{1,t-1} \\ \epsilon_{2,t-1} \end{bmatrix}$$

Or shorter

$$\mathbf{X}_t - \phi \mathbf{X}_{t-1} = \epsilon_t - \theta \epsilon_{t-1} \tag{2}$$

The one-step prediction

$$\begin{aligned} \mathbf{X}_{t+1} - \phi \mathbf{X}_t &= \epsilon_{t+1} - \theta \epsilon_t \qquad \Rightarrow \\ \hat{\mathbf{X}}_{t+1|t} &= E[\mathbf{X}_{t+1} | X_t, X_{t-1}, \dots] \\ &= \phi \mathbf{X}_t - \theta \epsilon_t \end{aligned}$$

In order to be able to predict $\hat{\mathbf{X}}_{t+1}$ we must know ϵ_t. An estimate of ϵ_t can be obtained by taking a few steps back in the time series and here set $\epsilon_t = 0$. We can then predict $\mathbf{X}$ one step ahead and the prediction error can be used as ϵ_t at the next prediction and so forth. By observing how the influence of previous values "dies out" we can determine how many steps back we need to take. A rewriting of (1) to an AR-form shows that $\det(\theta(B)) = 1 - 0.4B$. I.e. the influence of previous X-values decays as 0.4^k $(1/\det(\theta(B)))$. Since $0.4^3 \simeq 0.06$ it is sufficient to go 3 steps back.

$$
\begin{aligned}
t-3: \qquad \epsilon'_{t-3} &= \mathbf{0} \Rightarrow \\
\hat{\mathbf{X}}_{t-2|t-3} &= \phi \mathbf{X}_{t-3} - \theta \cdot \mathbf{0} \\
&= \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 2.9 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 1.24 \\ 2.42 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
t-2: \qquad \epsilon'_{t-2} &= \mathbf{X}_{t-2} - \hat{\mathbf{X}}_{t-2|t-3} = \begin{bmatrix} -1.04 \\ 2.48 \end{bmatrix} \\
\hat{\mathbf{X}}_{t-1|t-2} &= \phi \mathbf{X}_{t-2} - \theta \cdot \epsilon'_{t-2} \\
&= \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} 0.2 \\ 4.9 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} -1.04 \\ 2.48 \end{bmatrix} \\
&= \begin{bmatrix} -2.78 \\ 2.52 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
t-1: \qquad \epsilon'_{t-1} &= \mathbf{X}_{t-1} - \hat{\mathbf{X}}_{t-1|t-2} = \begin{bmatrix} 0.48 \\ 0.98 \end{bmatrix} \\
\hat{\mathbf{X}}_{t|t-1} &= \phi \mathbf{X}_{t-1} - \theta \cdot \epsilon'_{t-1} \\
&= \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} -2.3 \\ 3.5 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.48 \\ 0.98 \end{bmatrix} \\
&= \begin{bmatrix} -3.94 \\ 1.14 \end{bmatrix}
\end{aligned}
$$

I.e.

$$
\epsilon'_t = \mathbf{X}_t - \hat{\mathbf{X}}_{t|t-1} = \begin{bmatrix} 0.54 \\ 0.66 \end{bmatrix}
$$

The one-step prediction from week t is then

$$
\begin{aligned}
\hat{\mathbf{X}}_{t+1|t} &= \phi \mathbf{X}_t - \theta \epsilon'_t \\
&= \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} -3.4 \\ 1.8 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0.4 \end{bmatrix} \begin{bmatrix} 0.54 \\ 0.66 \end{bmatrix} = \begin{bmatrix} -3.80 \\ -0.36 \end{bmatrix}
\end{aligned}
$$

The two-step prediction from week t is

$$
\begin{aligned}
\hat{\mathbf{X}}_{t+2|t} &= E[\mathbf{X}_{t+2} | \mathbf{X}_t, \mathbf{X}_{t+1}, \dots] \\
&= \phi \hat{\mathbf{X}}_{t+1|t} = \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.7 \end{bmatrix} \begin{bmatrix} -3.80 \\ -0.36 \end{bmatrix} = \begin{bmatrix} -2.82 \\ -1.77 \end{bmatrix}
\end{aligned}
$$

The variance is found by rewriting equation (2) into MA-form.

$$
\begin{aligned}
(2) \Rightarrow \quad & (\mathbf{I} - \phi B)\mathbf{X}_t = (\mathbf{I} - \theta B)\epsilon_t \qquad (\mathbf{I} \text{ is the unity matrix}) \\
\Downarrow \quad & \mathbf{X}_t = (\mathbf{I} - \phi B)^{-1}(\mathbf{I} - \theta B)\epsilon_t \\
\Updownarrow \quad & \mathbf{X}_t = (\mathbf{I} + \phi B + \phi^2 B^2 + \cdots)(\mathbf{I} - \theta B)\epsilon_t \\
\Updownarrow \quad & \mathbf{X}_t = (\mathbf{I} + (\phi - \theta)B + \phi(\phi - \theta)B^2 + \phi^2(\phi - \theta)B^3 + \cdots)\epsilon_t
\end{aligned}
$$

I.e. $\psi_1 = (\phi - \theta)$, which leads to the following estimates of the variances

$$
\mathbf{V}[1] = \mathbf{\Sigma} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}
$$

$$
\begin{aligned}
\mathbf{V}[2] &= \mathbf{\Sigma} + \psi_1 \mathbf{\Sigma} \psi_1^\top \\
&= \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 0.8 & -0.6 \\ 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0.8 & 0.4 \\ -0.6 & 0.3 \end{bmatrix} \\
&= \begin{bmatrix} 4.96 & -0.72 \\ -0.72 & 2.26 \end{bmatrix}
\end{aligned}
$$

Question 2.

We introduce the notation

$$
\mathbf{X}_{t+1} | \mathbf{X}_t, \mathbf{X}_{t-1}, \cdots = \mathbf{X}_{t+1} | \mathcal{X}_t = \left(\begin{array}{c} X_{1t+1} \\ X_{2t+1} \end{array} \middle| \mathcal{X}_t \right)
$$

Furthermore we assume that $\mathbf{X}_{t+1}|\mathcal{X}_t$ is normal distributed[1]. Previously $E[\mathbf{X}_{t+1}|\mathcal{X}_t] = \hat{\mathbf{X}}_{t+1|t}$ and $V[\mathbf{X}_{t+1}|\mathcal{X}_t] = V[1]$ were calculated. We wish to determine

$$
E[X_{1t+1}|X_{2t+1}, \mathcal{X}_t] \text{ and } V[X_{1t+1}|X_{2t+1}, \mathcal{X}_t]
$$

[1]The assumption about the normal distribution is not essential since the formulas for the conditional expectation and the conditional variance holds, provided that the conditional expectation is assumed to be linear in X_2.

They can be determined using Theorem 10.1 in Chapter 10 in the book.

$$
\begin{aligned}
E[X_{1t+1}|X_{2t+1}, \mathcal{X}_t] = {} & E[X_{1t+1}|\mathcal{X}_t]+ \\
& \operatorname{Cov}[X_{1t+1}|X_{2t+1}, \mathcal{X}_t]V^{-1}[X_{2t+1}|\mathcal{X}_t](X_{2t+1} - E[X_{2t+1}|\mathcal{X}]) \\
& = -3.80 + (-1)\frac{1}{2}(-1.20 - (-0.36)) = -3.38 \\
V[X_{1t+1}|X_{2t+1}, \mathcal{X}_t] = {} & V[X_{1t+1}|\mathcal{X}_t]- \\
& \operatorname{Cov}[X_{1t+1}|X_{2t+1}, \mathcal{X}_t]V^{-1}[X_{2t+1}|\mathcal{X}_t]\operatorname{Cov}[X_{1t+1}|X_{2t+1}, \mathcal{X}_t] \\
& = 2 - (-1)\frac{1}{2}(-1) = 1.5
\end{aligned}
$$

Solution 10.3

Question 1.

The system described by the system equation

$$X_t = X_{t-1} + e_t$$

Since $\{e_t\}$ is white noise $\{X_t\}$ is a random walk (equation (5.126))

Question 2.

For the given system and observation equation $C = A = 1$, $V_1 = 1$ and $V_2 = 2$. Since $\Sigma_{0|0}^{XX} = 1$

$$\begin{aligned}
\Sigma_{1|0}^{xx} &= 1 + 1 = 2 \\
\Sigma_{1|0}^{yy} &= 1 \cdot 2 \cdot 1 + 2 = 4 \\
K(1) &= 2 \cdot 1 \cdot \frac{1}{4} = \frac{1}{2} \\
\Sigma_{1|1}^{xx} &= 2 - \frac{1}{2} \cdot 4 \cdot \frac{1}{2} = 1 = \Sigma_{XX}(0|0)
\end{aligned}$$

In general we get

$$\begin{aligned}
\Sigma_{t|t}^{xx} &= 1 \\
\Sigma_{t+1|t}^{xx} &= 2 \\
\Sigma_{t+1|t}^{yy} &= 4 \\
K(t) &= \frac{1}{2}
\end{aligned}$$

Which leads to

$$\hat{X}_{t+1|t} = \hat{X}_{t|t-1} + \frac{1}{2}\left(Y_t - \hat{X}_{t|t-1}\right) \tag{3}$$

Question 3.

Equation (3) can be written as

$$\begin{aligned}\hat{X}_{t+1|t} &= \frac{1}{2}\left(Y_t + \hat{X}_{t|t-1}\right) \\ &= \frac{1}{2}Y_t + \left(\frac{1}{2}\right)^2 Y_{t-1} + \left(\frac{1}{2}\right)^2 \hat{X}_{t-1|t-2} \\ &= \sum_{j=1}^{t}\left(\frac{1}{2}\right)^j Y_{t+1-j} + \left(\frac{1}{2}\right)^t \hat{X}_{0|0}\end{aligned}$$

which is seen to be exponential smoothing.

Solution 10.4

Question 1.

$$
\begin{aligned}
&(1-\phi_1 B)(1-\phi_4 B^4)X_t = (1-\theta b)\epsilon_t \qquad \Leftrightarrow \\
&X_t - \phi_1 X_{t-1} - \phi_4 X_{t-4} + \phi_1\phi_4 X_{t-5} = \epsilon_t - \theta\epsilon_{t-1}
\end{aligned}
$$

On state space form

$$
\begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \end{bmatrix}
=
\begin{bmatrix} \phi_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \phi_4 & 0 & 0 & 0 & 1 \\ -\phi_1\phi_4 & 0 & 0 & 0 & 0 \end{bmatrix}
\begin{bmatrix} X_1(t-1) \\ X_2(t-1) \\ X_3(t-1) \\ X_4(t-1) \\ X_5(t-1) \end{bmatrix}
+
\begin{bmatrix} 1 \\ -\theta \\ 0 \\ 0 \\ 0 \end{bmatrix} \epsilon_t
$$

$$
X(t) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \\ X_3(t) \\ X_4(t) \\ X_5(t) \end{bmatrix}
$$

Question 2.

$$
\left.\begin{aligned}
X_1(t) &= -\phi_1 X_1(t-1) + X_2(t-1) + e(t) \\
X_2(t) &= -\phi_2 X_1(t-1) - \theta_1 e(t)
\end{aligned}\right\} \Rightarrow
$$

$$
X_1(t) = -\phi_1 X_1(t-1) - \phi_2 X_1(t-2) + e(t) - \theta_1 e(t-1)
$$

Since

$$
Y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = X_1(t)
$$

we get

$$
Y(t) + \phi_1 Y(t-1) + \phi_2 Y(t-2) = e(t) - \theta_1 e(t)
$$

I.e. $\{Y(t)\}$ can be described by an ARMA(2,1)-process.

Question 3.

Determination of the auto-covariance function of $\{X_1(t)\}$

$$\begin{aligned} X_1(t) = 0.8X_1(t-1) + e(t) \quad \Rightarrow \\ X_1(t)X_1(t-k) = 0.8X_1(t-1)X_1(t-k) + e(t)X_1(t-1) \end{aligned} \tag{4}$$

Taking the expected values leads to

$$\gamma_{X_1}(k) = 0.8\gamma_{X_1}(k-1) \qquad (k \geq 1) \tag{5}$$

Furthermore by multiplying (4) by $X_1(t)$ we get

$$\begin{aligned} X_1(t)X_1(t) = 0.8X_1(t-1)X_1(t) + e(t)X_1(t) \quad \Rightarrow \\ \gamma_{X_1}(0) = 0.8\gamma_{X_1}(1) + \sigma_e^2 \end{aligned} \tag{6}$$

By using (5) for $k = 1$ we get

$$\gamma_{X_1}(0) = \frac{1}{1-0.8^2}\sigma_e^2 = \frac{\sigma_e^2}{0.36} \tag{7}$$

and

$$\gamma_{X_1}(k) = \frac{0.8^k}{0.36}\sigma_e^2 \tag{8}$$

Determination of the auto-covariance function of $\{Y(t)\}$

$$\begin{aligned} \gamma_Y(k) &= \mathrm{Cov}[Y(t), Y(t+k)] \\ &= \mathrm{Cov}[X_1(t) + v(t), X_t(t+k) + v(t+k)] \\ &= \begin{cases} \gamma_{X_1}(0) + \sigma_v^2 & (k = 0) \\ \gamma_{X_1}(k) & (k \neq 0) \end{cases} \end{aligned} \tag{9}$$

From equation (8) and (9) we get the auto-correlation

$$\begin{aligned} \rho_Y(k) &= \frac{\gamma_Y(k)}{\gamma_Y(0)} \\ &= \begin{cases} \frac{\frac{0.8^k}{0.36}\sigma_e^2}{\frac{\sigma_e^2}{0.36}+\sigma_v^2} = \frac{0.8^k}{1+0.36\frac{\sigma_v^2}{\sigma_e^2}} & (k \neq 0) \\ 1 & (k \neq 0) \end{cases} \end{aligned}$$

In figure 1 the auto-correlation function is plotted for $\sigma_v^2/\sigma_e^2 = 0$ and for $\sigma_v^2/\sigma_e^2 = 1$ in figure 2. In the latter case it is observed that the exponential sequence is not starting in 1, but from a point further down the y-axis. The placement of the point depends on the relationship between the variance of the measurement noise σ_e^2 and the variance of the process noise σ_v^2. In figure 3 the auto-correlation is shown for $\sigma_v^2/\sigma_e^2 = \infty$

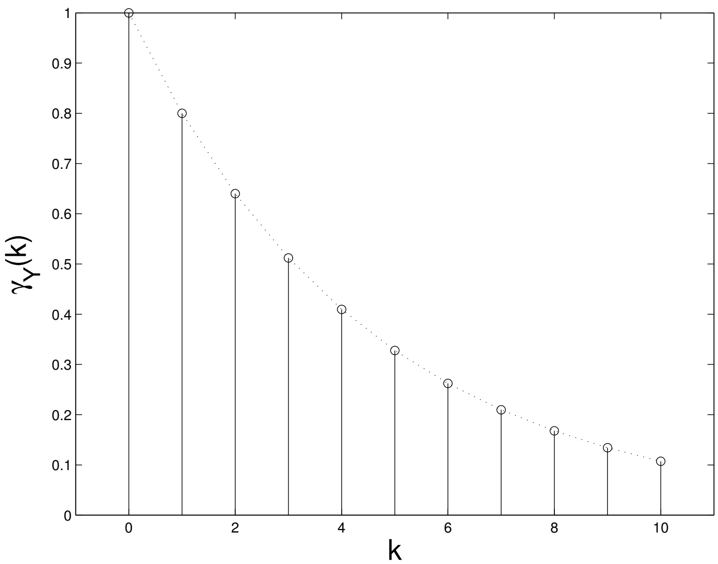

Figure 1: Auto-correlation function for $\sigma_v^2/\sigma_e^2 = 0$.

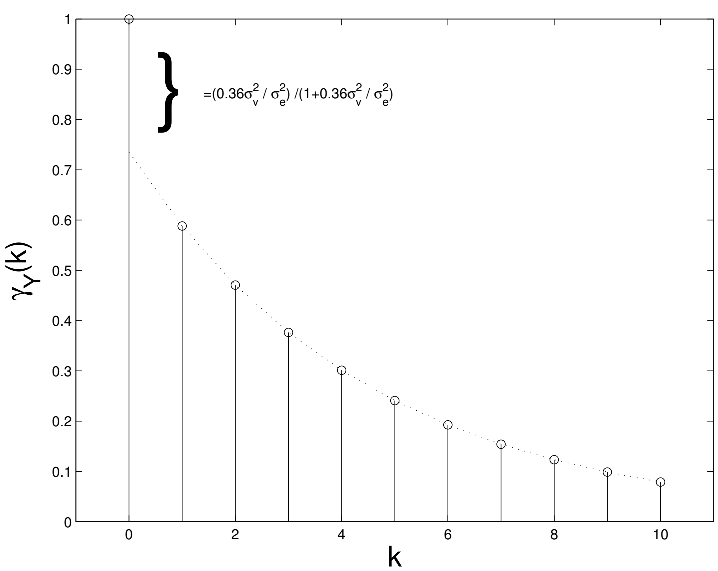

Figure 2: Auto-correlation function for $\sigma_v^2/\sigma_e^2 = 1$.

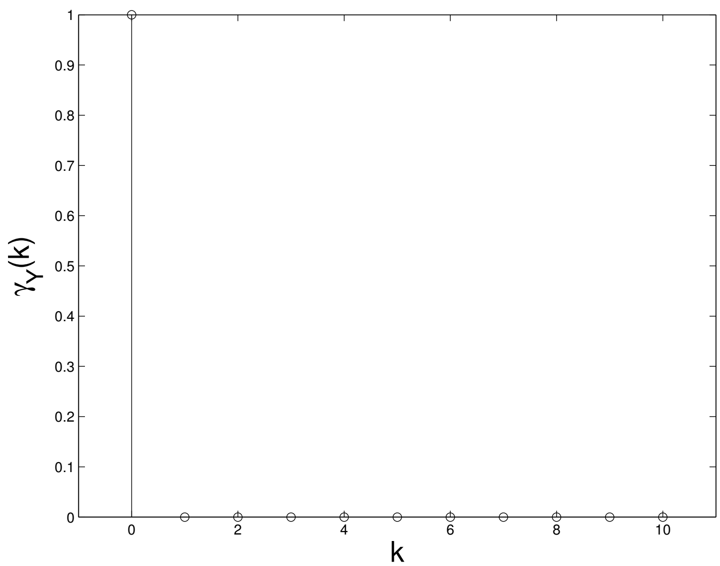

Figure 3: Auto-correlation function for $\sigma_v^2/\sigma_e^2 = \infty$.