Time Series Analysis

Solutions to problems in Chapter 9

IMM

Solution 9.1

Question 1.

$$\begin{aligned}
X_{1,t} &= \alpha_{11}X_{1,t-1} + \alpha_{12}X_{2,t-1} + \epsilon_{1,t} \Rightarrow \\
X_{1,t-k}X_{1,t} &= \alpha_{11}X_{1,t-k}X_{1,t-1} + \alpha_{12}X_{1,t-k}X_{2,t-1} + X_{1,t-k}\epsilon_{1,t} \Rightarrow \\
\mathrm{E}[X_{1,t-k}X_{1,t}] &= \alpha_{11}\mathrm{E}[X_{1,t-k}X_{1,t-1}] + \alpha_{12}\mathrm{E}[X_{1,t-k}X_{2,t-1}] + \mathrm{E}[X_{1,t-k}\epsilon_{1,t}] \Rightarrow \\
\gamma_{11}(k) &= \underline{\underline{\alpha_{11}\gamma_{11}(k-1) + \alpha_{12}\gamma_{12}(k-1)}} \qquad (k \geq 1)
\end{aligned}$$

$$\begin{aligned}
X_{2,t-k}X_{1,t} &= \alpha_{11}X_{2,t-k}X_{1,t-1} + \alpha_{12}X_{2,t-k}X_{2,t-1} + X_{2,t-k}\epsilon_{1,t} \Rightarrow \\
\gamma_{21}(k) &= \underline{\underline{\alpha_{11}\gamma_{21}(k-1) + \alpha_{12}\gamma_{22}(k-1)}} \qquad (k \geq 1)
\end{aligned}$$

$$\begin{aligned}
X_{1,t-k}X_{2,t} &= \alpha_{21}X_{1,t-k}X_{1,t-1} + \alpha_{22}X_{1,t-k}X_{2,t-1} + X_{1,t-k}\epsilon_{2,t} \Rightarrow \\
\gamma_{12}(k) &= \underline{\underline{\alpha_{21}\gamma_{11}(k-1) + \alpha_{22}\gamma_{12}(k-1)}} \qquad (k \geq 1)
\end{aligned}$$

$$\begin{aligned}
X_{2,t-k}X_{2,t} &= \alpha_{21}X_{2,t-k}X_{1,t-1} + \alpha_{22}X_{2,t-k}X_{2,t-1} + X_{2,t-k}\epsilon_{2,t} \Rightarrow \\
\gamma_{22}(k) &= \underline{\underline{\alpha_{21}\gamma_{21}(k-1) + \alpha_{22}\gamma_{22}(k-1)}} \qquad (k \geq 1)
\end{aligned}$$

which on matrix form can be written as

$$\begin{bmatrix} \gamma_{11}(k) & \gamma_{21}(k) \\ \gamma_{12}(k) & \gamma_{22}(k) \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix} \begin{bmatrix} \gamma_{11}(k-1) & \gamma_{21}(k-1) \\ \gamma_{12}(k-1) & \gamma_{22}(k-1) \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} \gamma_{11}(k) & \gamma_{21}(k) \\ \gamma_{12}(k) & \gamma_{22}(k) \end{bmatrix} = \begin{bmatrix} \gamma_{11}(k-1) & \gamma_{12}(k-1) \\ \gamma_{21}(k-1) & \gamma_{22}(k-1) \end{bmatrix} \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \qquad (k \geq 1)$$

Estimation of the intial values $\gamma_{11}(0), \gamma_{12}(0) = \gamma_{21}(0)$ and $\gamma_{22}(0)$

$$\begin{aligned}
X_{1,t} &= \alpha_{11}X_{1,t-1} + \alpha_{12}X_{2,t-1} + \epsilon_{1,t} \Rightarrow \\
X_{1,t}^2 &= (\alpha_{11}X_{1,t-1} + \alpha_{12}X_{2,t-1} + \epsilon_{1,t})^2 \Rightarrow \\
X_{1,t}X_{1,t} &= \alpha_{11}^2 X_{1,t-1}X_{1,t-1} + \alpha_{12}^2 X_{2,t-1}X_{2,t-1} + 2\alpha_{11}\alpha_{12}X_{1,t-1}X_{2,t-1} \\
&\quad + \alpha_{11}X_{1,t-1}\epsilon_{1,t} + \alpha_{12}X_{2,t-1}\epsilon_{1,t} + \epsilon_{1,t}^2 \Rightarrow \\
\gamma_{11}(0) &= \underline{\underline{\alpha_{11}^2\gamma_{11}(0) + \alpha_{12}^2\gamma_{22}(0) + 2\alpha_{11}\alpha_{12}\gamma_{12}(0) + \sigma_1^2}}
\end{aligned}$$

Using the same procedure

$$\gamma_{22}(0) = \underline{\underline{\alpha_{21}^2\gamma_{11}(0) + \alpha_{22}^2\gamma_{22}(0) + 2\alpha_{21}\alpha_{22}\gamma_{12}(0) + \sigma_2^2}}$$

$$\gamma_{12}(0) = \underline{\underline{\alpha_{11}\alpha_{21}\gamma_{11}(0) + \alpha_{12}\alpha_{22}\gamma_{22}(0) + (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\gamma_{12}(0) + \sigma_{12}^2}}$$

The last equation is obtained by multiplying the two difference expressions for $X_{1,t}$ and $X_{2,t}$ and taking the expected values.

Question 2.

The initial values are found by inserting the values $\alpha_{11} = 0.6$, $\alpha_{12} = -0.5$, $\alpha_{21} = 0.4$ and $\alpha_{22}\alpha_{11}\alpha_{22} = 0.5$ together with $\sigma_1^1 = \sigma_2^2 = 1$ and $\sigma_{12}^2 = 0$ in the above equations. We get

$$\begin{aligned}
0.64\gamma_{11}(0) - 0.25\gamma_{22}(0) + 0.6\gamma_{12}(0) &= 1 \\
-0.16\gamma_{11}(0) + 0.75\gamma_{22}(0) - 0.4\gamma_{12}(0) &= 1 \\
0.24\gamma_{11}(0) - 0.25\gamma_{22}(0) - 0.9\gamma_{12}(0) &= 0
\end{aligned}$$

The solution is

$$\gamma_{11}(0) = \frac{\begin{vmatrix} 1 & -0.25 & 0.6 \\ 1 & 0.75 & -0.4 \\ 0 & -0.25 & -0.9 \end{vmatrix}}{\begin{vmatrix} 0.64 & -0.25 & 0.6 \\ -0.16 & 0.75 & -0.4 \\ 0.24 & -0.25 & -0.9 \end{vmatrix}} = \frac{-1.15}{-0.52} = \underline{\underline{2.21}}$$

$$\gamma_{22}(0) = \frac{\begin{vmatrix} 0.64 & 1 & 0.6 \\ -0.16 & 1 & -0.4 \\ 0.24 & 0 & -0.9 \end{vmatrix}}{\begin{vmatrix} 0.64 & -0.25 & 0.6 \\ -0.16 & 0.75 & -0.4 \\ 0.24 & -0.25 & -0.9 \end{vmatrix}} = \frac{-0.96}{-0.52} = \underline{\underline{1.85}}$$

$$\gamma_{12}(0) = \frac{\begin{vmatrix} 0.64 & -0.25 & 1 \\ -0.16 & 0.75 & 1 \\ 0.24 & -0.25 & 0 \end{vmatrix}}{\begin{vmatrix} 0.64 & -0.25 & 0.6 \\ -0.16 & 0.75 & -0.4 \\ 0.24 & -0.25 & -0.9 \end{vmatrix}} = \frac{-0.04}{-0.52} = \underline{\underline{0.08}}$$

Using the recursion formula found in the previous question the covariance functions can be calculated

$$k = 1: \quad \begin{bmatrix} \gamma_{11}(1) & \gamma_{21}(1) \\ \gamma_{12}(1) & \gamma_{22}(1) \end{bmatrix} = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 2.21 & 0.08 \\ 0.08 & 1.85 \end{bmatrix} = \begin{bmatrix} 1.29 & -0.88 \\ 0.92 & 0.96 \end{bmatrix}$$

$$k = 2: \quad \begin{bmatrix} \gamma_{11}(2) & \gamma_{21}(2) \\ \gamma_{12}(2) & \gamma_{22}(2) \end{bmatrix} = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 1.29 & -0.88 \\ 0.92 & 0.96 \end{bmatrix} = \begin{bmatrix} 0.31 & -1.01 \\ 0.98 & 0.13 \end{bmatrix}$$

$$k = 3: \quad \begin{bmatrix} \gamma_{11}(3) & \gamma_{21}(3) \\ \gamma_{12}(3) & \gamma_{22}(3) \end{bmatrix} = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0.31 & -1.01 \\ 0.98 & 0.13 \end{bmatrix} = \begin{bmatrix} -0.30 & -0.67 \\ 0.61 & -0.34 \end{bmatrix}$$

$$k = 4: \quad \begin{bmatrix} \gamma_{11}(4) & \gamma_{21}(4) \\ \gamma_{12}(4) & \gamma_{22}(4) \end{bmatrix} = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} -0.30 & -0.67 \\ 0.61 & -0.34 \end{bmatrix} = \begin{bmatrix} -0.49 & -0.23 \\ 0.19 & -0.44 \end{bmatrix}$$

$$k = 5: \quad \begin{bmatrix} \gamma_{11}(5) & \gamma_{21}(5) \\ \gamma_{12}(5) & \gamma_{22}(5) \end{bmatrix} = \begin{bmatrix} 0.6 & -0.5 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} -0.49 & -0.23 \\ 0.19 & -0.44 \end{bmatrix} = \begin{bmatrix} -0.39 & 0.08 \\ -0.10 & -0.31 \end{bmatrix}$$

The auto and cross correlation functions can now be determined from the covariance functions. The result is summarized in Table 1. The auto and cross correlation functions are plotted in Figure 1 and 2. When plotting the cross correlation function we have utilized that $\rho_{12}(-k) = \rho_{21}(k)$.

k	0	1	2	3	4	5
$\rho_{11}(k)$	1	0.58	0.14	-0.14	-0.22	-0.18
$\rho_{22}(k)$	1	0.52	0.07	-0.18	-0.24	-0.17
$\rho_{12}(k)$	0.04	0.45	0.48	0.30	0.09	-0.05
$\rho_{21}(k)$	0.04	-0.44	-0.50	-0.33	-0.11	0.04

Table 1: The auto and cross correlation functions

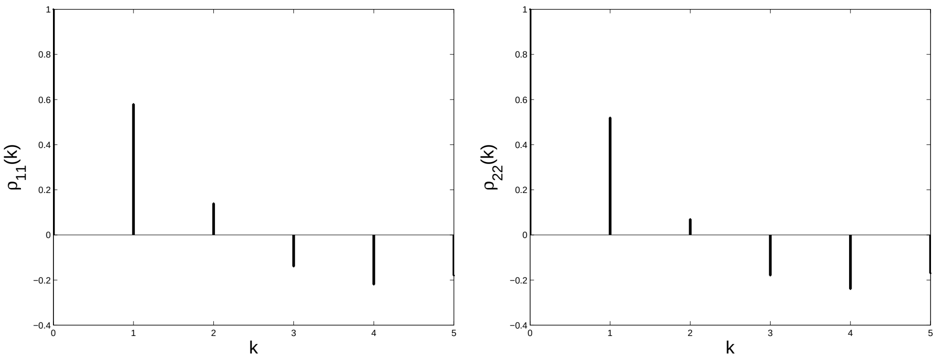

Figure 1: Auto correlation functions

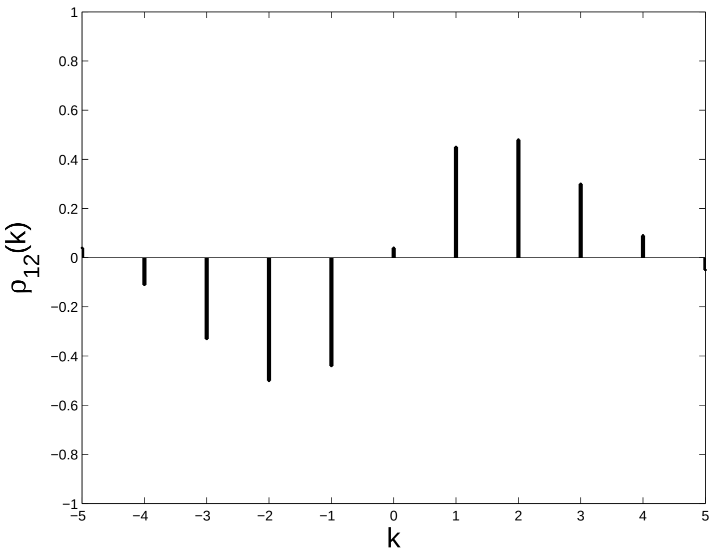

Figure 2: Cross correlation function

Solution 9.2

Question 1.

Given

$$\begin{bmatrix} X_{t,1} \\ X_{t,2} \end{bmatrix} = \begin{bmatrix} -1.0 & -2.5 \\ 1.0 & 2.0 \end{bmatrix} \begin{bmatrix} X_{1,t-1} \\ X_{2,t-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{bmatrix}$$

or in matrix notation

$$\mathbf{X}_t = -\boldsymbol{\phi}\mathbf{X}_{t-1} + \boldsymbol{\epsilon}_t \tag{1}$$

The covariance matrix function is

$$\Gamma_k = \begin{bmatrix} \gamma_{11}(k) & \gamma_{12}(k) \\ \gamma_{21}(k) & \gamma_{22}(k) \end{bmatrix} = \mathrm{Cov}[X_t, X_{t+k}] = \mathrm{E}[X_t X_{t+k}^\top]$$

By multiplying (1) with $\mathbf{X}_{t-k}^\top$ we get

$$\mathbf{X}_t\mathbf{X}_{t-k}^\top = -\boldsymbol{\phi}\mathbf{X}_{t-1}\mathbf{X}_{t-k}^\top + \boldsymbol{\epsilon}_t\mathbf{X}_{t-k}^\top$$

By taking the expected values we get

$$\begin{aligned} \boldsymbol{\Gamma}_{-k} &= -\,\boldsymbol{\phi}\boldsymbol{\Gamma}_{-(k-1)} + 0 \Rightarrow \\ \boldsymbol{\Gamma}_{-k}^\top &= -\,\boldsymbol{\Gamma}_{-(k-1)}^\top\boldsymbol{\phi}^\top \end{aligned}$$

As $\boldsymbol{\Gamma} = \mathrm{E}[\mathbf{X}_t\mathbf{X}_{t+k}^\top] = \mathrm{E}[(\mathbf{X}_{t+k}^\top\mathbf{X}_t)^\top] = \mathrm{E}[(\mathbf{X}_t^\top\mathbf{X}_{t-k})^\top] = \boldsymbol{\Gamma}_{-k}^\top$ the expression becomes

$$\underline{\underline{\boldsymbol{\Gamma}_k = -\boldsymbol{\Gamma}_{k-1}\boldsymbol{\phi}^\top}}\,, \tag{2}$$

The initial value $\boldsymbol{\Gamma}_0$ is found by multiplying with $\mathbf{X}_t^\top$, i.e.

$$\begin{aligned} \mathbf{X}_t\mathbf{X}_t^\top =& (\boldsymbol{\phi}\mathbf{X}_{t-1} - \boldsymbol{\epsilon}_t)(\boldsymbol{\phi}\mathbf{X}_{t-1} - \boldsymbol{\epsilon}_t)^\top \Rightarrow \\ \mathbf{X}_t\mathbf{X}_t^\top =& \boldsymbol{\phi}\mathbf{X}_{t-1}\mathbf{X}_{t-1}^\top\boldsymbol{\phi}^\top + \boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top + \boldsymbol{\phi}\mathbf{X}_{t-1}\boldsymbol{\epsilon}_t^\top + \boldsymbol{\epsilon}_t\mathbf{X}_{t-1}\boldsymbol{\phi}^\top \end{aligned}$$

by taking the expected value we get

$$\underline{\underline{\boldsymbol{\Gamma}_0 = \boldsymbol{\phi}\boldsymbol{\Gamma}_0\boldsymbol{\phi}^\top + \boldsymbol{\Sigma}}} \tag{3}$$

Question 2.

Inserting the known values for $\mathbf{\Sigma}$ and $\boldsymbol{\phi}$ in (3) we obtain ($\gamma_{12}(0) = \gamma_{21}(0)$)

$$\begin{bmatrix} \gamma_{11}(0) & \gamma_{12}(0) \\ \gamma_{21}(0) & \gamma_{22}(0) \end{bmatrix} = \begin{bmatrix} -1.0 & -2.5 \\ 1.0 & 2.0 \end{bmatrix} \begin{bmatrix} \gamma_{11}(0) & \gamma_{12}(0) \\ \gamma_{21}(0) & \gamma_{22}(0) \end{bmatrix} \begin{bmatrix} -1.0 & 1.0 \\ -2.5 & 2.0 \end{bmatrix} + \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix}$$

which have the solutions

$$(\gamma_{11}(0), \gamma_{22}(0), \gamma_{12}(0)) = (32.6, 16.4, -20.7)$$

From the recursive formula (2) the auto and cross covariance functions can now be determined as

$$\mathbf{\Gamma}_k = \mathbf{\Gamma}_0(\boldsymbol{\phi}^\top)^k$$

and from here the auto and cross correlation functions are calculated. The result is summarized in Table 2.

k	0	1	2	3	4	5	6	7	8	9	10
$\rho_{11}(k)$	1	0.59	0.09	-0.21	-0.25	-0.15	-0.02	0.05	0.06	0.04	0.01
$\rho_{21}(k)$	-0.90	-0.38	0.06	0.26	0.21	0.10	-0.02	-0.06	-0.06	-0.02	0.0
$\rho_{12}(k)$	-0.90	-0.88	-0.43	0.01	0.21	0.22	0.11	-0.01	-0.06	-0.05	-0.03
$\rho_{22}(k)$	1	0.74	0.23	-0.13	-0.25	-0.18	-0.06	0.03	0.06	0.05	0.01

Table 2: The auto and cross correlation functions

The auto and cross correlation functions are plotted in Figure 3 and 4. When plotting the cross correlation function we have utilized that $\rho_{12}(-k) = \rho_{21}(k)$.

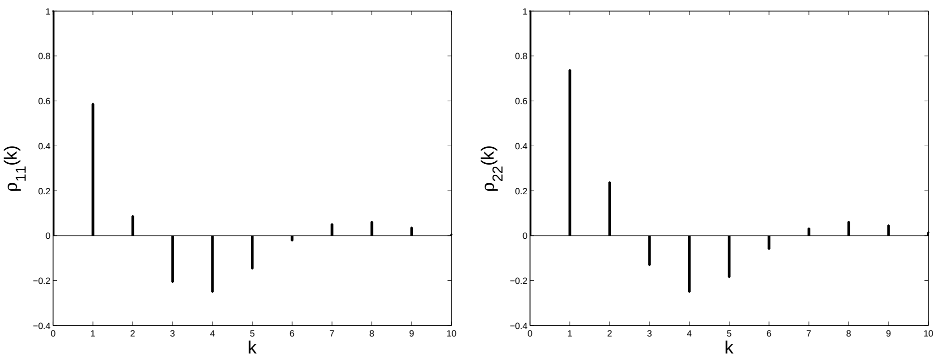

Figure 3: Auto correlation functions

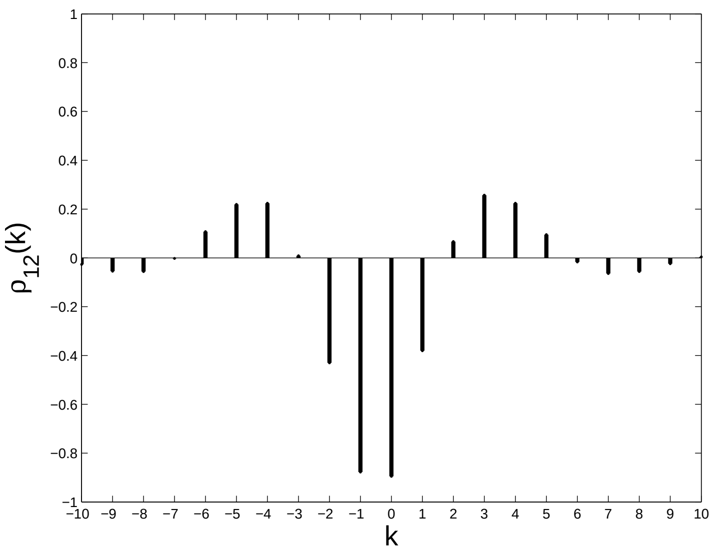

Figure 4: Cross correlation function

Question 3.

The impulse respons function for the transfer from $\epsilon_{1,t}$ to $X_{2,t}$ is found by sending a "1" through the system, i.e.

$$\boldsymbol{\epsilon}_t = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \text{for} \quad t = 0 \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \text{for} \quad t \neq 0 \end{cases}$$

The result is shown in Table 3 and it is plotted in Figure 5.

k	0	1	2	3	4	5	6	7	8	9	10
$X_{1,t}$	1	-1	1.5	-1	-0.25	0.25	0.375	-0.25	-0.0625	0.0625	0.09
$X_{2,t}$	0	1	1	0.5	0	-0.25	0.25	0.125	0	0.0625	0.0625

Table 3: Impusle response function

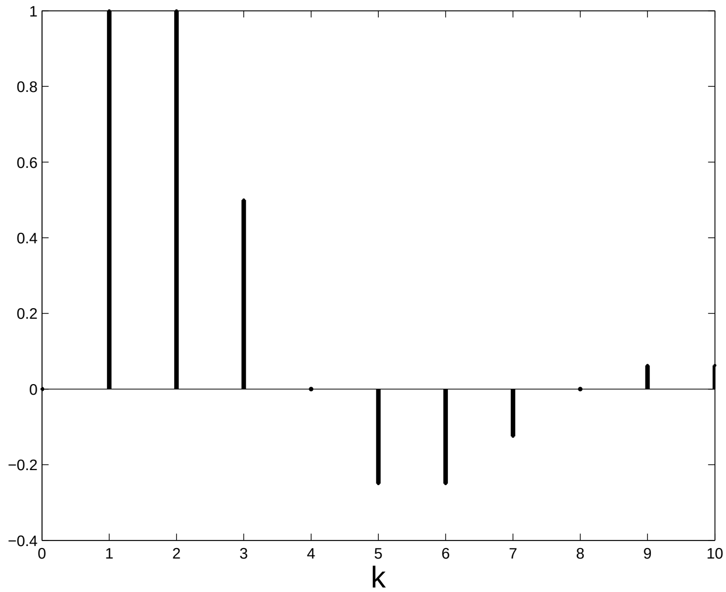

Figure 5: Impulse response function from $\epsilon_{1,t}$ to $X_{2,t}$

Solution 9.3

Firstly the cross-correlation coefficients are examined to see if there is an indication of relation between the processes.

The number of observations is N=120. Since there has been carried out a prewhitening of the input series the following holds

$$V[r_{\alpha\beta}(k)] \simeq \frac{1}{N}$$

I.e. the $2 \cdot \sigma$-limits for test of the hypothesis $\rho_{xy}(k) = 0$ are $\pm 2/\sqrt{N} = \pm 2/\sqrt{120} = \pm 0.183$. From the supplied values of $r_{\alpha\beta}(k)$ it is seen that the hypothesis must be rejected for $\rho_{xy}(0)$ and $\rho_{xy}(1)$ for the given foundation. Therefore a relation between the input and the output series must be expected.

Question 1.

An estimate of the impulse response function h_k can be found by

$$\hat{h}_k = \frac{C_{\alpha\beta}(k)}{\hat{\sigma}_\alpha^2} = \frac{r_{\alpha\beta}(k)\hat{\sigma}_\alpha\hat{\sigma}_\beta}{\hat{\sigma}_\alpha^2} = r_{\alpha\beta}(k)\frac{\hat{\sigma}_\beta}{\hat{\sigma}_\alpha}$$

which leads to the following estimates of the impulse respons function (table 4 and figure 6).

k	0	1	2	3	4	5	6	7	8	9	10
$\hat{h}_k$	0.141	0.148	0.056	0.043	-0.024	0.043	-0.023	0.029	-0.023	0.061	0.029

Table 4: Estimated impulse response function

Question 2.

If we assume that only $\hat{h}_0$ and $\hat{h}_1$ are an expression for a connection we get the model

$$\begin{aligned} Y_t &= h_0 X_t + h_1 X_{t-1} \text{ or} \\ Y_t &= (\omega_0 + \omega_1 B) X_t \end{aligned}$$

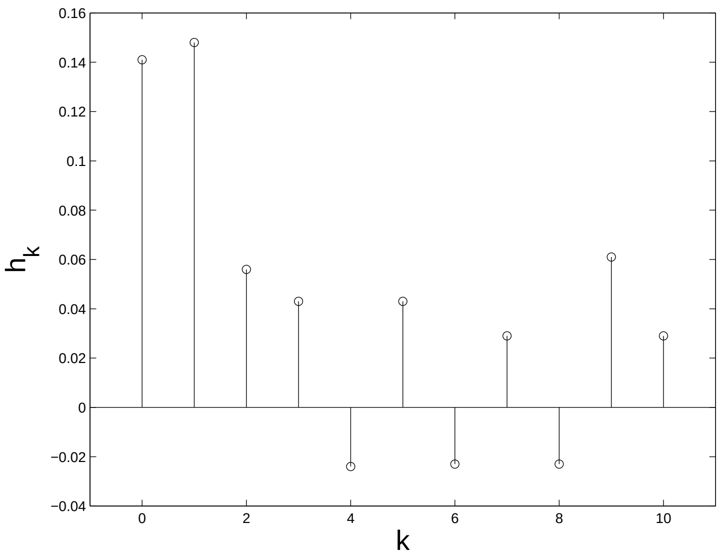

Figure 6: Estimated impulse response function.

The estimates of the impulse response function for lag 2 and 3 can alternatively be an expression of an exponential decay starting in lag 1. Since we cannot ignore $\hat{k}_0$ the model is

$$Y_t = \frac{\omega_0 + \omega_1 B}{1 + \hat{\delta} B} X_t$$

Question 3.

Since

$$\frac{\hat{\omega}_0 + \hat{\omega}_1 B}{1 + \hat{\delta} B} = (\hat{\omega}_0 + \hat{\omega}_1 B)(1 - \hat{\delta} B + \hat{\delta}^2 B^2 - \dots) = \hat{h}_0 + \hat{h}_1 B + \hat{h}_2 B^2 + \dots \Rightarrow$$
$$\hat{\omega}_0 + (\hat{\omega}_1 - \hat{\omega}_0 \hat{\delta}) B + (-\hat{\omega}_1 \hat{\delta} + \hat{\omega}_0 \hat{\delta}^2) B^2 = \hat{h}_0 + \hat{h}_1 B + \hat{h}_2 B^2 + \dots$$

A comparison of the coefficients in front of $B^j, j = 0, 1, 2$ leads to

$$\begin{aligned} \hat{\omega}_0 &= \hat{h}_0 & \Rightarrow \quad & \hat{\omega}_0 = 0.141 \\ -\hat{\delta}\hat{h}_1 &= \hat{h}_2 & \Rightarrow \quad & \hat{\delta}_0 = -0.378 \\ \hat{\omega}_1 &= \hat{h}_1 + \hat{\delta}\hat{h}_0 & \Rightarrow \quad & \hat{\omega}_1 = 0.095 \end{aligned}$$

Question 4.

If X_t also depends on Y_t we must use a bivariate model.

Solution 9.4

Question 1.

$$Y_t = X_{2,t} = -\theta_{21} B\epsilon_{1,t} + (1 - \theta_{22}B)\epsilon_{2,t}$$

The auto-covariance function of Y_t

$$\begin{aligned}
\gamma_Y(k) &= \mathrm{Cov}[Y_t, Y_{t+k}] \\
&= \mathrm{Cov}[-\theta_{21}\epsilon_{1,t-1} + \epsilon_{2,t} - \theta_{22}\epsilon_{2,t-1}, -\theta_{21}\epsilon_{1,t+k-1} + \epsilon_{2,t+k} - \theta_{22}\epsilon_{2,t+k-1}] \\
&= \begin{cases} \theta_{21}^2\sigma_1^2 + (1+\theta_{22}^2)\sigma_2^2 + 2\theta_{21}\theta_{22}\sigma_{12} & k = 0 \\ -\theta_{21}\sigma_{12}^2 - \theta_{22}\sigma_2^2 & k = \pm 1 \\ 0 & |k| \geq 2 \end{cases}
\end{aligned}$$

From this it can be seen that Y_t can be described by a MA(1)-process. Since the MA(1)-process can be written as

$$Y_t = \epsilon_t^\star - \theta^\star \epsilon_{t-1}^\star = (1 - \theta^\star B)\epsilon_t^\star$$

with the corresponding auto-covariance function

$$\gamma_Y(k) = \begin{cases} (1+\theta^{\star 2})\sigma_\epsilon^{\star 2} & k = 0 \\ -\theta^\star \sigma_\epsilon^{\star 2} & k = \pm 1 \\ 0 & |k| \geq 2 \end{cases}$$

I.e. $\theta^\star$ and $\sigma_\epsilon^\star$ can be found be solving the equations

$$\begin{aligned}
(1+\theta^{\star 2})\sigma_\epsilon^{\star 2} &= \theta_{21}^2\sigma_1^2 + (1+\theta_{22}^2)\sigma_2^2 + 2\theta_{21}\theta_{22}\sigma_{12} \\
-\theta^\star \sigma_\epsilon^{\star 2} &= -\theta_{21}\sigma_{12}^2 - \theta_{22}\sigma_2^2
\end{aligned}$$

Question 2.

$$\begin{bmatrix} 1-\phi_{11}B & -\phi_{12}B \\ -\phi_{21}B & 1-\phi_{22}B \end{bmatrix} \begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \frac{1}{(1-\phi_{11}B)(1-\phi_{22}B) - \phi_{21}\phi_{12}B^2} \begin{bmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix} \Leftrightarrow$$

$$\begin{bmatrix} X_{1t} \\ X_{2t} \end{bmatrix} = \frac{1}{1-(\phi_{11}+\phi_{22})B + (\phi_{11}\phi_{22} - \phi_{21}\phi_{12})B^2} \begin{bmatrix} 1-\phi_{22}B & \phi_{12}B \\ \phi_{21}B & 1-\phi_{11}B \end{bmatrix} \begin{bmatrix} \epsilon_{1t} \\ \epsilon_{2t} \end{bmatrix}$$

Since $Y_t = X_{2t}$

$$(1 - (\phi_{11} + \phi_{22})B + (\phi_{11}\phi_{22} - \phi_{21}\phi_{12})B^2)Y_t = \phi_{21}B\epsilon_{2t} + (1 - \phi_{11}B)\epsilon_{2t}$$

Using the same procedure as in question 1 the right hand side can be written as a MA(1)-process

$$(1 - (\phi_{11} + \phi_{22})B + (\phi_{11}\phi_{22} - \phi_{21}\phi_{12})B^2)Y_t = (1 - \theta' B)\epsilon'_t$$

Hence $\{Y_t\}$ can be described by an ARMA(2,1)-process.

Solution 9.5

Question 1.

From Theorem 9.3 we get

$$\begin{vmatrix} 1-2B & 2B \\ -1.25B & 1+B \end{vmatrix} = 0 \Leftrightarrow 1 - B + 0.5B^2 = 0$$

The roots of $z^2(1 - z^{-1} + 0.5z^{-2})$ are

$$z = \frac{1 \pm \sqrt{1-2}}{2} = 0.5 \pm 0.5i$$

As both roots are lying within the unit circle ($\sqrt{0.5^2 + 0.5^2} = 0.7 < 1$) the process is stationary.
From Theorem 9.4 we get

$$\begin{vmatrix} 1 & 0.5B \\ -0.5B & 1-B \end{vmatrix} = 0 \Leftrightarrow 1 - B + 0.25B^2 = 0$$

The roots of $z^2(1 - z^{-1} + 0.25z^{-2})$ are

$$z = \frac{1 \pm \sqrt{1-1}}{2} = 0.5 \ \text{(double root)}$$

As the root lye within the unit circle the process is invertible.

Question 2.

From Equation (9.38) we have the covariance matrix function for k=0

$$\begin{aligned}\boldsymbol{\Gamma}(0) =& \boldsymbol{\Sigma} + \boldsymbol{\theta}\boldsymbol{\Sigma}\boldsymbol{\theta}^\top \\ =& \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ -0.5 & -1 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 0.5 & -1 \end{bmatrix} \\ =& \begin{bmatrix} 3 & -3.5 \\ -3.5 & 6.5 \end{bmatrix}\end{aligned}$$

The covariance matrix functions for $k > 0$ can be obtained as

$$\begin{aligned}\boldsymbol{\Gamma}(1) =& \boldsymbol{\Sigma}\boldsymbol{\theta}^\top = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 0 & -0.5 \\ 0.5 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & -3 \end{bmatrix} \\ \boldsymbol{\Gamma}(k) =& 0 \ \text{ for } |k| \geq 2\end{aligned}$$

Appyling that $\rho_{12}(-k) = \rho_{21}(k)$ we get the following values for the cross correlation function $\rho_{12}(k)$.

$$\rho_{12}(k) = \begin{cases} \frac{\gamma_{21}(1)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} = & \frac{2}{\sqrt{3 \cdot 6.5}} = 0.46 & k = -1 \\ \frac{\gamma_{12}(0)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} = & \frac{-3.5}{\sqrt{3 \cdot 6.5}} = -0.79 & k = 0 \\ \frac{\gamma_{12}(1)}{\sqrt{\gamma_{11}(0)\gamma_{22}(0)}} = & \frac{1}{\sqrt{3 \cdot 6.5}} = 0.23 & k = 1 \\ & 0 & |k| \geq 2 \end{cases}$$

The cross correlation function is plotted in Figure 7.

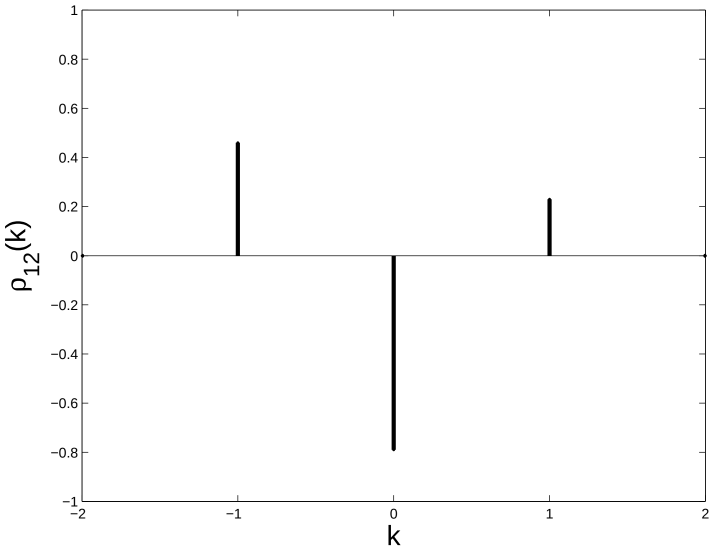

Figure 7: The cross correlation function $\rho_{12}(k)$

Solution 9.6

The $\boldsymbol{\psi}$ matrices can (according to Remark 9.2 on page 263) be found by sending a "1", the identity matrix, trough the system, i.e.

$$\boldsymbol{\epsilon}_t = \begin{cases} \mathbf{I} & \text{for} \quad t = 0 \\ \mathbf{0} & \text{for} \quad t \neq 0 \end{cases}$$

Given an ARMA$(\mathbf{P}, \mathbf{Q})$ process

$$\mathbf{Y}_t + \boldsymbol{\phi}_1 \mathbf{Y}_{t-1} + \cdots + \boldsymbol{\phi}_p \mathbf{Y}_{t-p} = \boldsymbol{\epsilon}_t + \boldsymbol{\theta}_1 \boldsymbol{\epsilon}_{t-1} + \cdots + \boldsymbol{\theta}_q \boldsymbol{\epsilon}_{t-q}$$

From where we get the following $\boldsymbol{\psi}$ matrices

$$\begin{aligned}
\boldsymbol{\psi}_0 =& \mathbf{Y}_0 = \boldsymbol{\epsilon}_0 = 1 \\
\boldsymbol{\psi}_1 =& \mathbf{Y}_1 = -\boldsymbol{\phi}_1 \mathbf{Y}_0 + \boldsymbol{\theta}_1 \boldsymbol{\epsilon}_{t-1} = -\boldsymbol{\phi}_1 + \boldsymbol{\theta}_1 \\
\boldsymbol{\psi}_2 =& \mathbf{Y}_2 = -\boldsymbol{\phi}_1 \mathbf{Y}_1 - \boldsymbol{\phi}_2 \mathbf{Y}_0 + \boldsymbol{\theta}_2 \boldsymbol{\epsilon}_{t-2} = -\boldsymbol{\phi}_1 \boldsymbol{\psi}_1 - \boldsymbol{\phi}_2 + \boldsymbol{\theta}_2 \\
&\vdots \\
\boldsymbol{\psi}_j =& \mathbf{Y}_j = -\boldsymbol{\phi}_1 \boldsymbol{\psi}_{j-1} - \boldsymbol{\phi}_2 \boldsymbol{\psi}_{j-2} - \cdots - \boldsymbol{\phi}_j + \boldsymbol{\theta}_j
\end{aligned}$$

Solution 9.7

We wish to derive equations for finding the parameter θ and the variance σ_ξ^2 such that

$$(1+\theta B)\xi_t = (1+\phi_{22}B)\epsilon_{1,t} - \phi_{12}\epsilon_{2,t}$$

We calculate the variance and covariance of the left-hand-side LHS and right-hand-side RHS of the above equation.

$$\begin{array}{ll}
\gamma(0): & \\
\text{LHS}: & \text{Cov}[\xi_t+\theta\xi_{t-1},[\xi_t+\theta\xi_{t-1}] = \sigma_\xi^2+\theta\sigma_\xi^2 = (1+\theta)\sigma_\xi^2 \\
\text{RHS}: & \text{Cov}[\epsilon_{1,t}+\phi_{22}\epsilon_{1,t-1}-\phi_{12}\epsilon_{2,t},\epsilon_{1,t}+\phi_{22}\epsilon_{1,t-1}-\phi_{12}\epsilon_{2,t}] = \\
 & \sigma_1^2+\phi_{22}\sigma_1^2-\phi_{12}\sigma_2^2 = (1+\phi_{22})\sigma_1^2+\phi_{12}\sigma_2^2 \\
\gamma(1): & \\
\text{LHS}: & \text{Cov}[\xi_t+\theta\xi_{t-1},[\xi_{t+1}+\theta\xi_t] = \theta\sigma_\xi^2 \\
\text{RHS}: & \text{Cov}[\epsilon_{1,t}+\phi_{22}\epsilon_{1,t-1}-\phi_{12}\epsilon_{2,t},\epsilon_{1,t+1}+\phi_{22}\epsilon_{1,t}-\phi_{12}\epsilon_{2,t+1}] = \phi_{22}\sigma_1^2-\phi_{12}\phi_{22}\sigma_{12}
\end{array}$$

The variance and covariance of the LHS and RHS must be identical, i.e. we have the following two equations with two unknown

$$\begin{aligned}
(1+\theta)\sigma_\xi^2 &= (1+\phi_{22})\sigma_1^2+\phi_{12}\sigma_2^2 \\
\theta\sigma_\xi^2 &= \phi_{22}\sigma_1^2-\phi_{12}\phi_{22}\sigma_{12}
\end{aligned}$$

From where we find the following solution

$$\theta = \underline{\underline{\frac{\phi_{22}\sigma_1^2-\phi_{12}\phi_{22}\sigma_{12}}{\sigma_1^2-\phi_{12}\sigma_2^2+\phi_{12}\phi_{22}\sigma_{12}}}}$$

$$\sigma_\xi^2 = \underline{\underline{\sigma_1^2-\phi_{12}\sigma_2^2+\phi_{12}\phi_{22}\sigma_{12}}}$$